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Non-critical string field theory for 2d quantum gravity coupled to (p,q)–conformal fields

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Abstract

We propose a non-critical string field theory for $2d$ quantum gravity coupled to (p,q) conformal fields. The Hamiltonian is described by the generators of the W_p algebra, and the Schwinger-Dyson equation is equivalent to a vacuum condition imposed on the generators of W_p algebra.

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1 Introduction

Non-critical string theories in dimensions $d < 1$, or equivalently, two-dimensional quantum gravity coupled to conformal theories with $c < 1$, have been intensively analyzed in the last eight years, either by the use of Liouville field theory or matrix models. The gravity aspects of the theories which refer to metric properties have been difficult to handle by continuum methods. In the framework of dynamical triangulations important progress was made by the calculation of the *transfer matrix* [1]. It allowed the calculation of the two-loop amplitude as a function of the *geodesic distance*. In fact, the transfer matrix offered a natural Hamiltonian $\overline{\mathcal{H}}$ for the one-string propagation and the geodesic distance between the two string loops played the role of *proper time* in a Hamiltonian framework. In [2], [3] and [4] the Hamiltonian $\overline{\mathcal{H}}$ for the one-string propagation was generalized to a genuine string field theory which allowed the calculation of any string amplitude, in [2] from a formal continuum point of view, in [3] from a stochastic quantization point of view, and in [4] from the point of view of dynamical triangulations which provide an explicit regularization of the theory. In [4] it was in addition shown that there was universality: the details of the random graphs used in the dynamical triangulation were not important. The concept of a string field Hamiltonian in the case of (p, q) conformal matter coupled to two-dimensional quantum gravity was further developed in a series of papers [5, 6, 7, 8]. One outcome was that in the case of $(p, q) = (m, m + 1)$ conformal theories, i.e. the unitary theories, consistency demanded a proper time T of dimension $\dim T = 1/m$, if we define the world-sheet to have dimension 2. In particular, this formula shows that the dimension of proper time goes to zero as the central charge $c = 1-6/m(m+1)$ goes to one for $m \to \infty$. If we still assume that T can be identified with the geodesic distance this implies that the internal Hausdorff dimension $d_h \to \infty$.

The concept of geodesic distance in quantum gravity was further clarified in [9, 10], where it was shown how the dimension of geodesic distance can be extracted unambiguously from the two-point function by the use of dynamical triangulations in the case of unitary theories coupled to gravity. The two-point function could be calculated explicitly in the case of pure gravity, i.e. for a $(p, q) = (2, 3)$ theory, where $c = 0$. However, subsequent numerical simulations indicated that the geodesic distance has dimension $1/2$ for *all* conformal field theories with $0 \leq c \leq 1$ coupled quantum gravity [11, 12]. This suggests that the relation between T and geodesic distance in general might be more complicated than anticipated from the study of pure two-dimensional quantum gravity.

As a step towards the resolution of these problems we present here detailed study of the Hamiltonian used in string field theory (SFT). A number of new features appear which have an independent interest, and we will present these in the rest of this article.

2 Pure gravity

In this section we investigate the (2,3)-SFT, i.e. pure gravity, using a new mode expansion. In addition we will clarify the relation between the Schwinger-Dyson

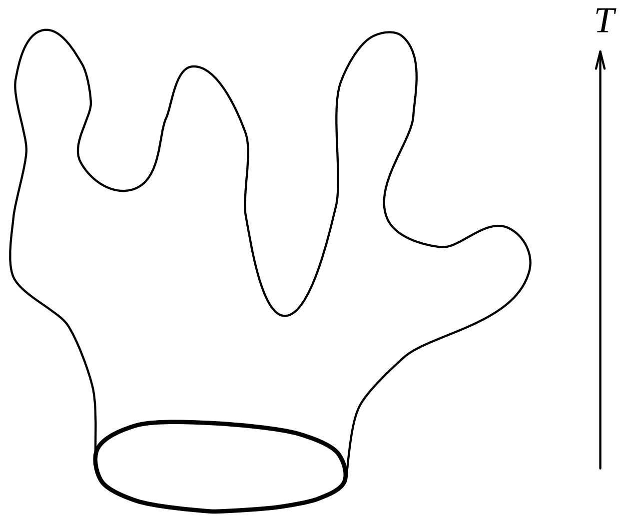

Figure 1: The propagation in proper time of a closed string which annihilates into the vacuum, allowing only disk topology.

equations in the string field theory and the vacuum conditions of the W_2 algebra.

2.1 String field theory for disk topology

Recall the string field theory as formulated in [2] in the case where the topology of the two-dimensional manifold is that of a sphere with boundaries (closed strings). Let $\Psi^\dagger(L)$ and $\Psi(L)$ be operators which creates and annihilates one closed string with length L, respectively. The commutation relation of the string operators are

$$[\Psi(L), \Psi^\dagger(L')] = \delta(L - L'), \tag{1}$$

$$[\Psi^\dagger(L), \Psi^\dagger(L')] = [\Psi(L), \Psi(L')] = 0. \tag{2}$$

In the Hamiltonian formalism the disk amplitude, $F_1^{(0)}(L;\mu)$, i.e. the amplitude that one closed string annihilates into the vacuum, is obtained by

$$F_1^{(0)}(L;\mu) \;=\; \lim_{T\to\infty} \langle \mathrm{vac}| e^{-T\mathcal{H}_{\mathrm{disk}}} \Psi^\dagger(L) |\mathrm{vac}\rangle, \tag{3}$$

where T is the so-called proper time, μ the cosmological constant and $\mathcal{H}_{\mathrm{disk}}$ the Hamiltonian for disk topology [2]. In Fig. 1 we show a typical configuration which contributes to the disk amplitude defined in (3). It should be noted that

$$\mathcal{H}_{\mathrm{disk}} |\mathrm{vac}\rangle \;=\; 0, \tag{4}$$

is a necessary condition in this formalism. The condition (4) means that a string is not created from the vacuum, i.e. the stability of the vacuum against decay into another physical state. The Laplace transformation of $F_1^{(0)}(L;\mu)$ is given by

$$\begin{aligned} F_1^{(0)}(\zeta;\mu) &= \int_0^\infty dL\, e^{-\zeta L} F_1^{(0)}(L;\mu) \\ &= \lambda(\zeta) \,+\, F_1^{\mathrm{univ}(0)}(\zeta;\mu)\,, \end{aligned} \tag{5}$$

where $\lambda(\zeta)$ and $F_1^{\rm univ(0)}(\zeta;\mu)$ are a divergent non-universal part and a universal part, respectively, given by,

$$\lambda(\zeta) = (\text{const.}) \times \varepsilon^{-3/2} \; - \; (\text{const.}) \times \varepsilon^{-1/2}\zeta \, , \tag{6}$$

$$F_1^{\rm univ(0)}(\zeta;\mu) = (\zeta - \frac{\sqrt{\mu}}{2})\sqrt{\zeta + \sqrt{\mu}} \, , \tag{7}$$

where ε is a cut off. In the context of dynamical triangulations we can view ε as the lattice spacing, i.e. the link length in the triangulations.

As already mentioned the above formalism is somewhat singular, and if we try to express $\mathcal{H}_{\rm disk}$ in terms of $\Psi(L)$ and $\Psi^\dagger(L)$ even the simplest commutators will need a regularization. For example, $\int_0^L dL' F_1^{(0)}(L';\mu) F_1^{(0)}(L-L';\mu)$, which is obtained from $[\mathcal{H}_{\rm disk}, \Psi^\dagger(L)]$, suffers from such a divergence. This problem was analyzed carefully in [4], where it was shown that a subtraction of a singular part of the Laplace transform of string wave function $\Psi^\dagger(L)$ would render the expressions finite. We therefore introduce the following Laplace transformed wave functions, $\Phi^\dagger(\zeta)$ and $\Psi(\eta)$,

$$\Phi^\dagger(\zeta) \;=\; \int_0^\infty dL \, e^{-L\zeta} \Psi^\dagger(L) \;-\; \lambda(\zeta) \, , \qquad \Psi(\eta) \;=\; \int_0^\infty dL \, e^{-L\eta} \Psi(L) \, . \tag{8}$$

The commutation relation of the wave functions are

$$[\, \Psi(\eta) \, , \, \Phi^\dagger(\zeta) \,] \;=\; \delta(\eta,\zeta), \qquad [\, \Phi^\dagger(\zeta_1) \, , \, \Phi^\dagger(\zeta_2) \,] \;=\; [\, \Psi(\eta_1) \, , \, \Psi(\eta_2) \,] \;=\; 0, \tag{9}$$

where $\delta(\eta,\zeta) = 1/(\eta+\zeta)$ is the Laplace transformation of $\delta(L-L')$. Then, the universal part of the disk amplitude is given by

$$F_1^{\rm univ(0)}(\zeta;\mu) \;=\; \lim_{T\to\infty} \langle {\rm vac} | e^{-T\mathcal{H}_{\rm disk}} \Phi^\dagger(\zeta) | {\rm vac} \rangle . \tag{10}$$

The actual form of the Hamiltonian can be derived from a careful study of a set of Schwinger-Dyson equations, also called the loop equations, in the context of dynamical triangulations [4]. In Fig. 2 we have shown a "typical" triangulation with a closed boundary and the topology of the disk. This is a concrete realization of the surface shown in Fig. 1. In the next subsection we will use such triangulations, with more general topology, to derive a string field Hamiltonian. By this procedure every step is well defined and the lattice spacing ε can be taken to zero in an unambiguous way.

However, it is possible to determine the continuum Hamiltonian from the following simple observations: one form of the loop equations (again derived from dynamical triangulations or matrix models), where the continuum limit is already taken is the following

$$0 \;=\; -\,\omega(\zeta,\mu) \,+\, \frac{\partial}{\partial\zeta} (F_1^{\rm univ(0)}(\zeta;\mu))^2, \tag{11}$$

where

$$\omega(\zeta,\mu) \;=\; 3\zeta^2 \;-\; \frac{3}{4}\mu. \tag{12}$$

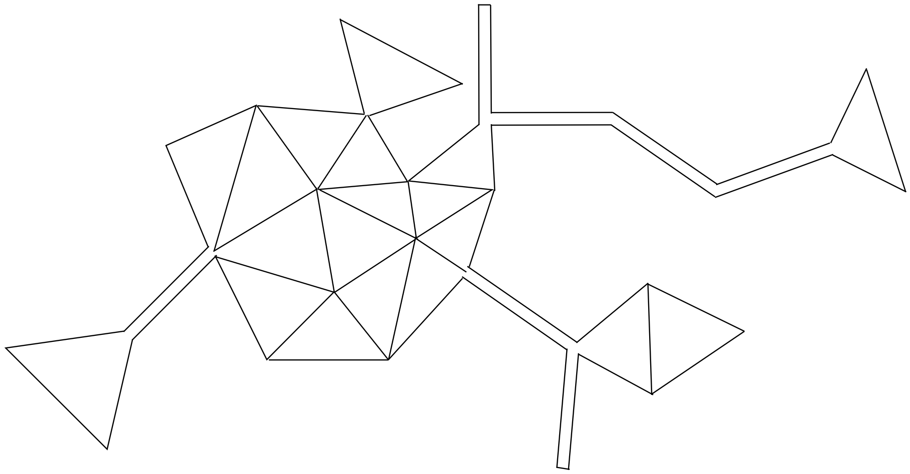

Figure 2: A "typical" triangulation with the topology of the "disk"

On the other hand, we can derive a kind of Schwinger-Dyson equation from Hamiltonian formalism if we assume that the limit (10) is smooth. Under this assumption we have

$$0 \;=\; -\lim_{T\to\infty}\frac{\partial}{\partial T}\langle\mathrm{vac}|e^{-T\mathcal{H}_{\mathrm{disk}}}\Phi^\dagger(\zeta)|\mathrm{vac}\rangle \;=\; \lim_{T\to\infty}\langle\mathrm{vac}|e^{-T\mathcal{H}_{\mathrm{disk}}}\mathcal{H}_{\mathrm{disk}}\Phi^\dagger(\zeta)|\mathrm{vac}\rangle\,. \quad (13)$$

Comparing (11) and (13) and using (10), one can expect

$$[\,\mathcal{H}_{\mathrm{disk}}\,,\,\Phi^\dagger(\zeta)\,] \;=\; -\,\omega(\zeta,\mu)\,+\,\frac{\partial}{\partial\zeta}(\Phi^\dagger(\zeta))^2, \quad (14)$$

if (4) is satisfied. Eq. (14) with the condition (4) leads to the well-defined Hamiltonian [4],

$$\mathcal{H}_{\mathrm{disk}} \;=\; \int_{-i\infty}^{+i\infty}\frac{d\zeta}{2\pi i}\Big\{-\,\omega(\zeta,\mu)\Psi(-\zeta)\,-\,(\Phi^\dagger(\zeta))^2\frac{\partial}{\partial\zeta}\Psi(-\zeta)\Big\}\,, \quad (15)$$

where we assume that $\Psi(\eta)|\mathrm{vac}\rangle = 0$. In the expression (15) regularization is unnecessary since any number of commutators of $\mathcal{H}_{\mathrm{disk}}$ with $\Phi^\dagger(\zeta)$ or $\Psi(\zeta)$ is finite due to the subtraction of $\lambda(\zeta)$ in (8).

After this brief review of known results we turn to the string mode expansion. The universal part of the disk amplitude, $F_1^{\mathrm{univ}(0)}(\zeta;\mu)$, has the following expansion around $\zeta = 0$,

$$F_1^{\mathrm{univ}(0)}(\zeta;\mu) \;=\; \zeta^{3/2} - \frac{3}{8}\mu\zeta^{-1/2} + \sum_{l=1,3,5,\ldots}\zeta^{-l/2-1}f_1^{(0)}(l;\mu)\,, \quad (16)$$

where $f_1^{(0)}(l;\mu)$ are known numbers. Therefore, we may assume that the string creation operators $\Phi^\dagger(\zeta)$ and $\Psi(\zeta)$ can be expanded as

$$\Phi^\dagger(\zeta) \;=\; \sum_{l=-5}^{\infty}\zeta^{-l/2-1}\phi_l^\dagger\,, \qquad \Psi(-\zeta) = \sum_{l=1}^{\infty}\zeta^{l/2}\phi_l\,, \quad (17)$$

where the creation and annihilation operators $\phi_l^\dagger$ and ϕ_l, $l \geq 1$, satisfy the commutation relation,

$$[\,\phi_l\,,\,\phi_{l'}^\dagger\,] \;=\; \delta_{l,l'}\,, \qquad [\,\phi_l^\dagger\,,\,\phi_{l'}^\dagger\,] \;=\; [\,\phi_l\,,\,\phi_{l'}\,] \;=\; 0\,. \tag{18}$$

These commutation relations reproduce (9). Substituting (17) into (14) we find

$$\phi_{-5}^\dagger = 1, \quad \phi_{-1}^\dagger = -\frac{3}{8}\mu, \quad \phi_{-4}^\dagger = \phi_{-3}^\dagger = \phi_{-2}^\dagger = \phi_0^\dagger = 0\,. \tag{19}$$

Somewhat surprising *it is not consistent to have* $\phi_{2n}^\dagger = 0$, as one would naively expect from the mode expansion (16) of the disk amplitude. Such a choice would force $\phi_{2n+1}^\dagger$ to be constants, in contradiction with the fact that $\Phi^\dagger(\zeta)$ is a non-trivial operator. Thus, we have

$$\Phi^\dagger(\zeta) \;=\; \zeta^{3/2} - \frac{3}{8}\mu\zeta^{-1/2} + \sum_{l=1}^{\infty}\zeta^{-l/2-1}\phi_l^\dagger\,. \tag{20}$$

Note that the operators $\phi_l^\dagger$, $l < 1$, which were introduced asymmetrically in (17), have to be constants, not genuine non-trivial operators.

The analogy of (10) is now

$$f_1^{(0)}(l;\mu) \;=\; \lim_{T\to\infty}\langle\mathrm{vac}|e^{-T\mathcal{H}_{\mathrm{disk}}}\phi_l^\dagger|\mathrm{vac}\rangle\,. \tag{21}$$

Since (21) should satisfy (16), $f_1^{(0)}(l;\mu) = 0$ for $l =$ even integer, i.e. $\phi_l^\dagger$ *with even* l *should be considered as a kind of null field* in the sense that any amplitude of the kind (21) where $\phi_l^\dagger$ is replaced by polynomials of $\phi_{l_i}^\dagger$'s will vanish in the $T \to \infty$ limit if only one l_i is even. Substituting (20) into (14), we find

$$[\,\mathcal{H}_{\mathrm{disk}}\,,\,\phi_l^\dagger\,] \;=\; -\,l\,(\,\frac{9}{128}\mu^2\delta_{l,2} \,+\, \phi_{l+1}^\dagger \,-\, \frac{3}{8}\theta_{l,4}\mu\phi_{l-3}^\dagger \,+\, \frac{1}{2}\theta_{l,6}\sum_{k=1}^{l-5}\phi_k^\dagger\phi_{l-k-4}^\dagger\,)\,, \tag{22}$$

where $\theta_{l,k} = 1$ for $l \geq k$ and $\theta_{l,k} = 0$ for $l < k$. Finally, we obtain from (22) and (4),

$$\mathcal{H}_{\mathrm{disk}} \;=\; -\,\frac{9}{64}\mu^2\phi_2 - \sum_{l=1}^{\infty}\phi_{l+1}^\dagger l\phi_l \,+\, \frac{3}{8}\mu\sum_{l=4}^{\infty}\phi_{l-3}^\dagger l\phi_l \,-\, \frac{1}{2}\sum_{l=6}^{\infty}\sum_{k=1}^{l-5}\phi_k^\dagger\phi_{l-k-4}^\dagger l\phi_l\,, \tag{23}$$

where we have used $\phi_l|\mathrm{vac}\rangle = 0$ for $l > 0$. The first term in (23) allows one string to vanishes into the vacuum, the second and the third terms are the "kinetic" part of $\mathcal{H}_{\mathrm{disk}}$, and the fourth term describes the splitting of one string into two strings.

2.2 String field theory for general topologies

Let us generalize the mode expression (23) for $\mathcal{H}_{\mathrm{disk}}$ to the Hamiltonian which produces general amplitudes for orientable surfaces. This Hamiltonian $\tilde{\mathcal{H}}$ has to contain an additional term which allows the merging of two strings into one. Further, we still want to maintain the stability of the vacuum, i.e.

$$\tilde{\mathcal{H}}|\mathrm{vac}\rangle \;=\; 0. \tag{24}$$

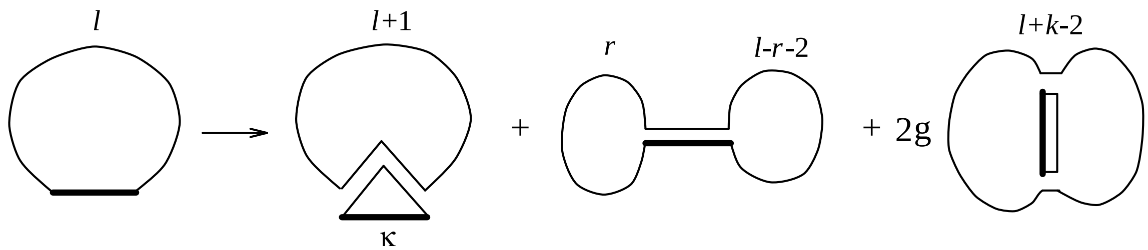

Figure 3: The possible deformations one can perform starting with a boundary link (shown as a thick line): One can remove a triangle if the link belongs to a triangle, or one can remove a double link, if the link is a part of such double link. The double link can occur in two situations shown symbolically on the figure: if one removes the double link, one boundary will be separated into two boundaries, or two boundaries will be merged into one boundary.

Let us shortly outline how we derive an expression for $\tilde{\mathcal{H}}$. As in the case for $\mathcal{H}_{\rm disk}$, the starting point is the special set of Schwinger-Dyson equations called the loop equations. Whatever Hamiltonian we derive, we want it to reproduce these equations via (13), only with $\tilde{\mathcal{H}}$ replacing $\mathcal{H}_{\rm disk}$. The loop equations have the graphical representation shown in Fig. 3. This figure expresses the possible change at the discretized level, using dynamical triangulations, when one deforms the boundary loop l. If we at the discretized level represent the boundary loop as $\Psi^\dagger(l)$, and denote by $\delta\Psi^\dagger(l)$ the change under a deformation of the boundary, the figure transforms into the following algebraic equation:

$$
\begin{aligned}
\langle \delta\Psi^\dagger(l) \prod_i \Psi^\dagger(k_i) \rangle \;=\; & \sum_{r=0}^{l-2} \Big(\langle \Psi^\dagger(r) \Psi^\dagger(l-r-2) \prod_i \Psi^\dagger(k_i) \rangle \\
& \qquad + \sum_S \langle \Psi^\dagger(r) \prod_{i\in S} \Psi^\dagger(k_i) \rangle \langle \Psi^\dagger(l-r-2) \prod_{j\in \bar{S}} \Psi^\dagger(k_j) \rangle \Big) \\
& + 2g \sum_j k_j \langle \Psi^\dagger(l+k_j-2) \prod_{i\neq j} \Psi^\dagger(k_i) \rangle \\
& + \kappa \langle \Psi^\dagger(l+1) \prod_i \Psi^\dagger(k_i) \rangle \;-\; \langle \Psi^\dagger(l) \prod_i \Psi^\dagger(k_i) \rangle \,, \qquad (25)
\end{aligned}
$$

for $l \geq 1$. In this formula S and $\bar{S}$ are partitions of the set of indices over which i runs, while g denotes the discretized version of the string coupling constant and κ is related to the cosmological constant as is clear from Fig. 3. In addition we have introduced the shorthand notation,

$$
\langle \;\; \equiv \;\; \langle {\rm vac} |\, e^{-T\tilde{\mathcal{H}}} \qquad \text{and} \qquad \rangle \;\; \equiv \;\; |{\rm vac}\rangle \,. \qquad (26)
$$

We here define $\Psi^\dagger(l=0) = 1$ in order to make the form of the algebraic equation (25) simple. Then, we get $\delta\Psi^\dagger(l=0) = 0$. By taking the discrete Laplace transformation of (25), we obtain

$$
\langle \delta\Psi^\dagger(x) \prod_i \Psi^\dagger(z_i) \rangle \;=\; x^2 \langle \Psi^\dagger(x) \Psi^\dagger(x) \prod_i \Psi^\dagger(z_i) \rangle
$$

$$
\begin{aligned}
&+ x^2 \sum_S \langle \Psi^\dagger(x) \prod_{i\in S} \Psi^\dagger(z_i)\rangle \langle \Psi^\dagger(x) \prod_{j\in \bar{S}} \Psi^\dagger(z_j)\rangle \\
&+ 2gx^2 \langle \Psi^\dagger(x)\Big(-x\frac{\partial}{\partial x}\Psi(\frac{1}{x})\Big) \prod_i \Psi^\dagger(z_i)\rangle \\
&+ \frac{\kappa}{x}\langle (\Psi^\dagger(x) - x\frac{\partial}{\partial x}\Psi^\dagger(x=0) - 1) \prod_i \Psi^\dagger(z_i)\rangle \\
&- \langle (\Psi^\dagger(x) - 1) \prod_i \Psi^\dagger(z_i)\rangle \,.
\end{aligned}
\tag{27}
$$

From this formula we conclude that

$$
\begin{aligned}
\delta\Psi^\dagger(x) \;&=\; x^2\Big(\Psi^\dagger(x)\Big)^2 \;+\; 2g\Big[x^2\Psi^\dagger(x)\Big(-x\frac{\partial}{\partial x}\Psi(\frac{1}{x})\Big)\Big]^{(+)} \\
&\quad + \frac{\kappa}{x}\Big(\Psi^\dagger(x) \;-\; x\frac{\partial}{\partial x}\Psi^\dagger(x=0) - 1\Big) - \;(\Psi^\dagger(x) - 1),
\end{aligned}
\tag{28}
$$

where $[x^n]^{(+)}$ means x^n if $n > 0$ and zero if $n \leq 0$.

The Schwinger-Dyson equation expresses that the expectation value of this deformation has to vanish for $T \to \infty$:

$$
\lim_{T\to\infty} \langle \delta\Psi^\dagger(l) \prod_i \Psi^\dagger(k_i)\rangle \;=\; 0, \tag{29}
$$

and string field theory emerges if we can find a $\tilde{\mathcal{H}}$ such that the vacuum condition is satisfied and [4]

$$
[\tilde{\mathcal{H}}, \Psi^\dagger(l)] \;=\; -l\delta\Psi^\dagger(l)\,. \tag{30}
$$

The Laplace transform of (30) is

$$
[\tilde{\mathcal{H}}, \Psi^\dagger(x)] \;=\; -x\frac{\partial}{\partial x}\,\delta\Psi^\dagger(x)\,, \tag{31}
$$

and it allows us to determine $\tilde{\mathcal{H}}$ up to a piece which is uniquely fixed by the vacuum condition. After taking the continuum limit of this equation, where

$$
\kappa = \kappa_c e^{-\varepsilon^2\mu}, \quad g = \varepsilon^5 G, \quad x = x_c e^{-\varepsilon\zeta}, \tag{32}
$$

define the continuum cosmological constant μ, the continuum string coupling constant G and our "continuum" Laplace transformation parameter ζ, we find the following continuum Hamiltonian [2, 3, 4]

$$
\tilde{\mathcal{H}} \;=\; \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i}\Big\{ -\omega(\zeta,\mu)\Psi(-\zeta) - (\Phi^\dagger(\zeta))^2\frac{\partial}{\partial\zeta}\Psi(-\zeta) - G\Phi^\dagger(\zeta)(\frac{\partial}{\partial\zeta}\Psi(-\zeta))^2\Big\}\,. \tag{33}
$$

We can now introduce a generating functional,

$$
Z_F^{\rm univ}[J] \;\stackrel{\rm def}{\equiv}\; \lim_{T\to\infty}\langle {\rm vac}|\, e^{-T\tilde{\mathcal{H}}}\, \exp[\int \frac{d\zeta}{2\pi i}\Phi^\dagger(\zeta)J(-\zeta)]|{\rm vac}\rangle\,, \tag{34}
$$

and the loop amplitudes are obtained from

$$\sum_{h=0}^{\infty} G^{h+N-1}\, F_N^{\mathrm{univ}(h)}(\zeta_1,\ldots,\zeta_N;\mu) \;=\; \frac{\delta^N}{\delta J(\zeta_1)\,\cdots\,\delta J(\zeta_N)} \ln Z_F^{\mathrm{univ}}[J]\bigg|_{J=0} . \tag{35}$$

Here $F_N^{\mathrm{univ}(h)}(\zeta_1,\ldots,\zeta_N;\mu)$ denotes the *universal* part of loop amplitudes,[1] i.e. the part which does not contain any cut off dependence. The cut off dependence appears only in $F_1^{(0)}(\zeta;\mu)$ through the function $\lambda(\zeta)$ and it has already been subtracted by the shift $\Psi^\dagger \to \Phi^\dagger$, as explained above.

The amplitudes $F_N^{\mathrm{univ}(h)}(\zeta_1,\ldots,\zeta_N;\mu)$ are in principle known from the loop equations coming from the matrix models and they allow expansions of the form,

$$\begin{aligned} F_N^{\mathrm{univ}(h)}(\zeta_1,\ldots,\zeta_N;\mu) \;&=\; \left(\Omega_1(\zeta_1)\delta_{N,1} + \Omega_2(\zeta_1,\zeta_2)\delta_{N,2}\right)\delta_{h,0} \\ &\quad + \sum_{l_i=1,3,5,\ldots} \zeta_1^{-l_1/2-1}\cdots\zeta_N^{-l_N/2-1} f_N^{(h)}(l_1,\ldots,l_N;\mu)\,, \end{aligned} \tag{36}$$

where

$$\Omega_1(\zeta) \;=\; \zeta^{3/2} - \frac{3}{8}\mu\zeta^{-1/2}\,, \tag{37}$$

$$\Omega_2(\zeta,\eta) \;=\; \frac{1}{4\sqrt{\zeta\eta}(\sqrt{\zeta}+\sqrt{\eta})^2}\,. \tag{38}$$

It is seen that $\Omega_1(\zeta)$ and $\Omega_2(\zeta,\eta)$ are precisely the parts of the universal loop amplitudes which fall off slower than $\zeta^{-3/2}$.

A set of "Hamiltonian" Schwinger-Dyson equations is obtained as above by assuming that the limit $T\to\infty$, which defines $Z_F^{\mathrm{univ}}[J]$, is smooth. This implies

$$0 \;=\; -\lim_{T\to\infty}\frac{\partial}{\partial T}\langle\mathrm{vac}|\, e^{-T\tilde{\mathcal{H}}}\, \exp\Big[\int\frac{d\zeta}{2\pi i}\Phi^\dagger(\zeta)J(-\zeta)\Big]\,|\mathrm{vac}\rangle\,. \tag{39}$$

The differentiation with respect to T brings down the operator $\tilde{\mathcal{H}}$, and we can move it outside the bra-vector as a functional differential operator $\tilde{\mathcal{H}}^\star$ in J,

$$\tilde{\mathcal{H}}^\star\, Z_F^{\mathrm{univ}}[J] \;=\; 0\,, \tag{40}$$

$$\tilde{\mathcal{H}}^\star \;=\; \int_{-i\infty}^{+i\infty}\frac{d\zeta}{2\pi i}\Big\{-\omega(\zeta,\mu)J(-\zeta) - \Big(\frac{\partial}{\partial\zeta}J(-\zeta)\Big)\Big(\frac{\delta}{\delta J(\zeta)}\Big)^2 - G\Big(\frac{\partial}{\partial\zeta}J(-\zeta)\Big)^2\frac{\delta}{\delta J(\zeta)}\Big\}\,. \tag{41}$$

This procedure can be made systematic by introducing the $\star$ operation as follows

$$\begin{aligned} (A_1A_2\cdots A_n)^\star \;&=\; A_n^\star\cdots A_2^\star A_1^\star\,, \\ (\Phi^\dagger(\zeta))^\star \;&=\; \frac{\delta}{\delta J(\zeta)}\,, \qquad (\Psi(\eta))^\star \;=\; J(\eta)\,, \\ (\langle\mathrm{vac}|)^\star &= |\mathrm{vac}\rangle, \qquad (|\mathrm{vac}\rangle)^\star = \langle\mathrm{vac}|. \end{aligned} \tag{42}$$

[1]There is some confusion concerning the use of *universal*. In some articles the functions $f_N^{(h)}$ to be defined below are denoted universal, while the amplitudes $F_N^{\mathrm{univ}(h)}$ are called non-universal even they are independent of the cut off.

Then, $\tilde{\mathcal{H}}^\star$ in (41) is obtained after applying the $\star$ operation to $\tilde{\mathcal{H}}$ in (33). The condition (24), which expresses the stability of the vacuum, becomes

$$\langle \mathrm{vac}|\tilde{\mathcal{H}}^\star \;=\; 0, \tag{43}$$

where $\langle \mathrm{vac}|J(\eta)=0$ is equivalent to $\Psi(\eta)|\mathrm{vac}\rangle = 0$.

Now, let us consider the string mode expansion. We have seen in the study of the disk amplitude that the *non-trivial operators* $\phi_l^\dagger$ were connected to the amplitudes $f_1^{(0)}(l;\mu)$, which have a Laplace transform which behaves as $\sum_{l=1,3,\ldots}\zeta^{-l/2-1}f_1^{(0)}(l;\mu)$. From (36)-(38) it follows that the Laplace transforms of $f_N^{(h)}(l_1,\ldots,l_N;\mu)$ have the same kind of expansion. Hence, it is natural to introduce a special generating function for the loop-functions $f_N^{(h)}(l_1,\ldots,l_N;\mu)$ and to expect that they have an representation analogous to (21) in terms of the fields ϕ_l and a Hamiltonian $\mathcal{H}$. In order to find $\mathcal{H}$ we first transform $\Omega_1(\zeta)$ and $\Omega_2(\zeta,\eta)$ away by writing

$$Z_F^{\mathrm{univ}}[J] = Z_\Omega[J]Z_f[j], \tag{44}$$

where

$$\ln Z_\Omega[J] \;\equiv\; \Omega[J] \;=\; \int\frac{d\zeta}{2\pi i}\Omega_1(\zeta)J(-\zeta)+\frac{G}{2}\int\frac{d\zeta}{2\pi i}\frac{d\eta}{2\pi i}\Omega_2(\zeta,\eta)J(-\zeta)J(-\eta). \tag{45}$$

With the definition we can write

$$\sum_{h=0}^{\infty}G^{h+N-1}\,f_N^{(h)}(l_1,\ldots,l_N;\mu) \;=\; \frac{\delta^N}{\delta j_{l_1}\,\cdots\,\delta j_{l_N}}\,\ln Z_f[j]\Big|_{j=0}, \tag{46}$$

where we have introduced the notation

$$\frac{\delta}{\delta J(\zeta)} \;=\; \Omega_1(\zeta)+\sum_{l=1}^{\infty}\zeta^{-l/2-1}\frac{\partial}{\partial j_l}, \qquad J(-\zeta)=\sum_{l=1}^{\infty}\zeta^{l/2}j_l\,. \tag{47}$$

The Hamiltonian $\tilde{\mathcal{H}}^\star$ acting on $Z_F^{\mathrm{univ}}[J]$ is naturally related to a Hamiltonian $\mathcal{H}^\star$ acting on $Z_f[j]$ by

$$\tilde{\mathcal{H}}^\star Z_F^{\mathrm{univ}}[J]=0 \qquad\Rightarrow\qquad \mathcal{H}^\star Z_f[j]=0, \tag{48}$$

where

$$\mathcal{H}^\star = e^{-\Omega[J]}\tilde{\mathcal{H}}^\star e^{\Omega[J]} = \tilde{\mathcal{H}}^\star - [\Omega[J],\tilde{\mathcal{H}}^\star]+\frac{1}{2}[\Omega[J],[\Omega[J],\tilde{\mathcal{H}}^\star]]\,. \tag{49}$$

The vacuum condition (24) and (43) are replaced by

$$\mathcal{H}|\mathrm{vac}\rangle \;=\; 0\,, \quad \text{and} \quad \langle\mathrm{vac}|\mathcal{H}^\star \;=\; 0\,, \tag{50}$$

where $\phi_l|\mathrm{vac}\rangle=0$ and $\langle\mathrm{vac}|j_l=0$ for any $l>0$. After some algebra we obtain

$$\begin{aligned}\mathcal{H}^\star \;=\; & -\frac{9}{64}\mu^2 j_2 - \frac{1}{8}Gj_4 + \frac{3}{16}G\mu j_1j_2 - \frac{1}{16}G^2j_1j_1j_2 \\ & -\sum_{l=1}^{\infty}lj_l\frac{\partial}{\partial j_{l+1}} + \frac{3}{8}\mu\sum_{l=4}^{\infty}lj_l\frac{\partial}{\partial j_{l-3}} \\ & -\frac{1}{2}\sum_{l=6}^{\infty}\sum_{k=1}^{l-5}lj_l\frac{\partial}{\partial j_k}\frac{\partial}{\partial j_{l-k-4}} - \frac{1}{8}G\sum_{k=1}^{\infty}\sum_{l=1}^{k+3}(k-l+4)j_{k-l+4}lj_l\frac{\partial}{\partial j_k}\,. \end{aligned} \tag{51}$$

If we write the star operation in terms of modes $\phi_l^\dagger$ and ϕ_l, it has the following realization

$$(\phi_l^\dagger)^\star \;=\; \frac{\partial}{\partial j_l}\,, \qquad (\phi_l)^\star \;=\; j_l\,. \tag{52}$$

It enables us to write $\mathcal{H}^\star$ in terms $\phi_l^\dagger$ and ϕ_l:

$$\begin{aligned}\mathcal{H} \;=\; & -\frac{9}{64}\mu^2\phi_2 - \frac{1}{8}G\phi_4 + \frac{3}{16}G\mu\phi_1\phi_2 - \frac{1}{16}G^2\phi_1\phi_1\phi_2 \\ & -\sum_{l=1}^{\infty}\phi_{l+1}^\dagger l\phi_l + \frac{3}{8}\mu\sum_{l=4}^{\infty}\phi_{l-3}^\dagger l\phi_l \\ & -\frac{1}{2}\sum_{l=6}^{\infty}\sum_{k=1}^{l-5}\phi_k^\dagger\phi_{l-k-4}^\dagger l\phi_l - \frac{1}{8}G\sum_{k=1}^{\infty}\sum_{l=1}^{k+3}\phi_k^\dagger(k-l+4)\phi_{k-l+4}l\phi_l\,. \end{aligned} \tag{53}$$

Note that $\mathcal{H}$ satisfies $\mathcal{H}_{\text{disk}} = \mathcal{H}|_{G=0}$.

We can finally verify that $Z_f[j]$ has the following representation in terms of $\mathcal{H}$ and the modes $\phi_l^\dagger$,

$$Z_f[j] \;=\; \lim_{T\to\infty}\langle\text{vac}|\, e^{-T\mathcal{H}}\, \exp(\sum_{l=1}^{\infty}\phi_l^\dagger j_l)\,|\text{vac}\rangle\,, \tag{54}$$

and in this way $\mathcal{H}$ *is* the Hamiltonian for the amplitudes $f_N^{(h)}$ in the same way as $\tilde{\mathcal{H}}$ is the Hamiltonian for the amplitudes $F_N^{\text{univ}(h)}$. In particular, we can derive $\mathcal{H}^\star Z_f[j] = 0$, i.e. the last equation in (48) from the analogy to (39):

$$0 \;=\; -\lim_{T\to\infty}\frac{\partial}{\partial T}\langle\text{vac}|\, e^{-T\mathcal{H}}\, \exp(\sum_{l=1}^{\infty}\phi_l^\dagger j_l)\,|\text{vac}\rangle\,. \tag{55}$$

2.3 Relation to W-algebras and τ-functions

The purpose of this subsection is to make the algebraic structure of $\mathcal{H}^\star$ more transparent. Recall the following formal representation of the so-called W-generators: Let the "current" $\alpha(z)$ be defined by

$$\alpha(z) = \sum_{n\in Z}\alpha_n z^{-n-1}\,, \tag{56}$$

where

$$\alpha_n \;=\; \begin{cases} -nx_{-n} & \text{if } n<0 \\ 0 & \text{if } n=0 \\ \frac{\partial}{\partial x_n} & \text{if } n>0 \end{cases}\,, \qquad [\alpha_m,\alpha_n] = m\delta_{m+n,0}\,. \tag{57}$$

The W-generators are defined by normal ordering with respect to the α_n generators by the formula,

$$W^{(k)}(z) \;=\; \sum_{l=0}^{k-1}\frac{(-1)^l}{(k-l)l!}\frac{([k-1]_l)^2}{[2k-2]_l}\Big(\frac{\partial}{\partial z}\Big)^l P^{(k-l)}(z)\,, \tag{58}$$

$$P^{(k)}(z) \;=\; :\Big(\frac{\partial}{\partial z}+\alpha(z)\Big)^k:\;1\,, \tag{59}$$

where $[k]_l = k(k-1)\cdots(k-l+1)$. The first few $W^{(k)}(z)$'s are

$$
\begin{aligned}
W^{(1)}(z) &= \alpha(z)\,, \\
W^{(2)}(z) &= \frac{1}{2} : \alpha(z)^2 :, \\
W^{(3)}(z) &= \frac{1}{3} : \alpha(z)^3 :, \\
W^{(4)}(z) &= \frac{1}{4} : \left\{ \alpha(z)^4 + \frac{2}{5}\alpha(z)\partial^2\alpha(z) - \frac{3}{5}(\partial\alpha(z))^2 \right\} :,
\end{aligned}
\tag{60}
$$

and each of the $W^{(k)}(z)$'s has a mode expansion,

$$
W^{(k)}(z) = \sum_{n\in Z} W_n^{(k)} z^{-n-k}, \tag{61}
$$

i.e. for the first few values of k,

$$
\begin{aligned}
W_n^{(1)} &= \alpha_n\,, \\
W_n^{(2)} &= \frac{1}{2} \sum_{a+b=n} : \alpha_a\alpha_b :, \\
W_n^{(3)} &= \frac{1}{3} \sum_{a+b+c=n} : \alpha_a\alpha_b\alpha_c :, \\
W_n^{(4)} &= \frac{1}{4} \sum_{a+b+c+d=n} : \alpha_a\alpha_b\alpha_c\alpha_d : \\
&\quad - \frac{1}{4} \sum_{a+b=n} \{(a+1)(b+1) - \frac{1}{5}(n+2)(n+3)\} : \alpha_a\alpha_b : \,.
\end{aligned}
\tag{62}
$$

Finally, the so-called p-reduced W-generators are obtained by only considering modes of $W^{(k)}$, $W^{(k-2)}$, ... with mode numbers which are multiples of p:

$$
\begin{aligned}
\overline{W}_n^{(1)} &= W_{pn}^{(1)}\,, \\
\overline{W}_n^{(2)} &= \frac{1}{p}\left\{ W_{pn}^{(2)} + \frac{1}{24}(p^2-1)\delta_{n,0} \right\}\,, \\
\overline{W}_n^{(3)} &= \frac{1}{p^2}\left\{ W_{pn}^{(3)} + \frac{1}{12}(p^2-1)W_{pn}^{(1)} \right\}\,, \\
\overline{W}_n^{(4)} &= \frac{1}{p^3}\left\{ W_{pn}^{(4)} + \frac{7}{20}(p^2-1)W_{pn}^{(2)} + \frac{7}{960}(p^2-1)^2\delta_{n,0} \right\}\,.
\end{aligned}
\tag{63}
$$

In case $p=2$, expressed in terms of the α_n's, we get for the *two-reduced* operators,

$$
\begin{aligned}
\overline{W}_n^{(1)} &= \alpha_{2n}\,, \\
\overline{W}_n^{(2)} &= \frac{1}{2}\left(\frac{1}{2} \sum_{a+b=2n} : \alpha_a\alpha_b : + \frac{1}{8}\delta_{n,0} \right), \\
\overline{W}_n^{(3)} &= \frac{1}{4}\left(\frac{1}{3} \sum_{a+b+c=2n} : \alpha_a\alpha_b\alpha_c : + \frac{1}{4}\alpha_{2n} \right).
\end{aligned}
\tag{64}
$$

We here also introduce the W-generators which do not depend on $\overline{W}_k^{(1)}$'s (or α_{pk}), i.e.,

$$\begin{aligned}
\overline{W}_n^{(2)\prime} &= \frac{1}{p}\left\{\frac{1}{2}\sum_{\substack{a\\(pn)}}\sum_{\substack{l\\(0 \bmod p)}} :\alpha_{a_1}^{[l_1]}\alpha_{a_2}^{[l_2]}: + \frac{1}{24}(p^2-1)\delta_{n,0}\right\}, \\
\overline{W}_n^{(3)\prime} &= \frac{1}{p^2}\left\{\frac{1}{3}\sum_{\substack{a\\(pn)}}\sum_{\substack{l\\(0 \bmod p)}} :\alpha_{a_1}^{[l_1]}\alpha_{a_2}^{[l_2]}\alpha_{a_3}^{[l_3]}:\right\}, \\
\overline{W}_n^{(4)\prime} &= \frac{1}{p^3}\left\{\frac{1}{4}\sum_{\substack{a\\(pn)}}\sum_{\substack{l\\(0 \bmod p)}} :\alpha_{a_1}^{[l_1]}\alpha_{a_2}^{[l_2]}\alpha_{a_3}^{[l_3]}\alpha_{a_4}^{[l_4]}: - \frac{p}{8}:\Big[\sum_{\substack{a\\(pn)}}\sum_{\substack{l\\(0 \bmod p)}}\alpha_{a_1}^{[l_1]}\alpha_{a_2}^{[l_2]}\Big]^2: \right.\\
&\qquad - \frac{1}{4}\sum_{\substack{a\\(pn)}}\sum_{\substack{l\\(0 \bmod p)}}\Big(\frac{1}{12}(p-6)(p^2-1)+l_1l_2\Big):\alpha_{a_1}^{[l_1]}\alpha_{a_2}^{[l_2]}: \\
&\qquad \left. - \frac{1}{5760}(p^2-1)(p-2)(p-3)(5p+7)\delta_{n,0}\right\},
\end{aligned} \tag{65}$$

where we have introduced the notation,

$$\begin{aligned}
\sum_{\substack{a\\(pn)}} &\stackrel{\text{def}}{\equiv} \sum_{a_1}\cdots\sum_{a_k} \quad \text{with} \quad \sum_{i=1}^{k} a_i = pn, \\
\sum_{\substack{l\\(0 \bmod p)}} &\stackrel{\text{def}}{\equiv} \sum_{l_1=1}^{p-1}\cdots\sum_{l_k=1}^{p-1} \quad \text{with} \quad \sum_{i=1}^{k} l_i = 0 \bmod p,
\end{aligned} \tag{66}$$

and

$$\alpha_n^{[i]} = \begin{cases} \alpha_n & (n = i \bmod p) \\ 0 & (\text{otherwise}) \end{cases}. \tag{67}$$

We now give the relation between the current $\alpha(z)$ and the string modes. We introduce the identification,

$$nx_n = \sqrt{G}\,nj_n + \frac{v_n}{\sqrt{G}}, \qquad \frac{\partial}{\partial x_n} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial j_n}. \tag{68}$$

Here v_n denotes a possible shift and depends on the cosmological constant μ. Then we can write

$$\alpha(z) = \frac{1}{\sqrt{G}}\Big(v(z) + \frac{\delta}{\delta j(z)}\Big) + \sqrt{G}\frac{\partial}{\partial z}\Big(zj(z)\Big), \tag{69}$$

and

$$\alpha^\star(z) = \frac{1}{\sqrt{G}}\Big(v(z) + \phi^\dagger(z)\Big) + \sqrt{G}\frac{\partial}{\partial z}\Big(z\phi(z)\Big), \tag{70}$$

where

$$\begin{aligned}
\frac{\delta}{\delta j(z)} &= \sum_{n=1}^{\infty}\frac{\partial}{\partial j_n}z^{-n-1}, \quad j(z) = \sum_{n=1}^{\infty} j_n z^{n-1}, \\
\phi^\dagger(z) &= \sum_{n=1}^{\infty}\phi_n^\dagger z^{-n-1}, \quad \phi(z) = \sum_{n=1}^{\infty}\phi_n z^{n-1}, \quad v(z) = \sum_{n=1}^{\infty} v_n z^{n-1}.
\end{aligned} \tag{71}$$

The mode expansion of the string field and its current, $\Phi^\dagger(\zeta)$ and $J(\zeta)$, are expressed by the modes $\phi^\dagger(z)$ and $j(z)$ if we identify $\zeta = z^p$, i.e. we write

$$\begin{aligned} \frac{\delta}{\delta J(\zeta)} &= \frac{1}{z^{p+k-2}}\Big(v(z) + \frac{\delta}{\delta j(z)}\Big), \quad J(-\zeta) \;=\; z^k j(z), \\ \Phi^\dagger(\zeta) &= \frac{1}{z^{p+k-2}}\Big(v(z) + \phi^\dagger(z)\Big), \quad \Psi(-\zeta) \;=\; z^k \phi(z), \end{aligned} \tag{72}$$

where k is integer and will be uniquely determined for each (p,q) model.

In the present case of pure gravity we have $k=1$, and

$$v_n \;=\; -\frac{3\mu}{8}\delta_{n,1} \,+\, \delta_{n,5}\,. \tag{73}$$

With these definitions the Hamiltonian can be rewritten in terms the two-reduced operators as

$$\mathcal{H}^\star \;=\; -2\sqrt{G}\,\overline{W}{}^{(3)}_{-2} + \frac{1}{2\sqrt{G}}\alpha_6 - \frac{3\mu}{8\sqrt{G}}\alpha_2\,. \tag{74}$$

The stability of the vacuum is expressed by (50). In the eq. (74) the second and the third terms are necessary and sufficient for the condition (50) to be satisfied. Recall the observation that $\phi^\dagger_l$ was a null field for l an even integer. By (52) and (68) this translates into the statement,

$$\overline{W}{}^{(1)}_n Z_f[j] = 0, \quad \text{or} \quad \frac{\partial}{\partial j_{2n}} Z_f[j] = 0, \tag{75}$$

for $n>0$. If we denote the part of $\overline{W}{}^{(2)}_n$ which does not depend on $\overline{W}{}^{(1)}_k$'s (or α_{2k}), $k>0$ by $\overline{W}{}^{(2)\prime}_n$, i.e.

$$\overline{W}{}^{(2)\prime}_n = \frac{1}{4}\Big(\sum_{l+m+1=n} :\alpha_{2l+1}\alpha_{2m+1}: + \frac{1}{4}\delta_{n,0}\Big)\,, \tag{76}$$

the Hamiltonian (74) can also rewritten as

$$\begin{aligned} \mathcal{H}^\star \;=\;& -2\sqrt{G}\,\Big(\frac{1}{12}\sum_{a+b+c=-2} :\overline{W}{}^{(1)}_a\overline{W}{}^{(1)}_b\overline{W}{}^{(1)}_c: + \sum_{a+b=-2}\overline{W}{}^{(1)}_a\overline{W}{}^{(2)\prime}_b\Big) \\ & + \frac{1}{2\sqrt{G}}\overline{W}{}^{(1)}_3 - \frac{3\mu}{8\sqrt{G}}\overline{W}{}^{(1)}_1\,, \end{aligned} \tag{77}$$

where since $\mathcal{H}^\star Z_f[j] = 0$ and $\overline{W}{}^{(1)}_n Z_f[j] = 0$ for $n>0$ (see (75)) this implies that $\overline{W}{}^{(2)\prime}_n Z_f[j] = 0$ for $n \geq -1$. These two conditions are generally believed to be sufficient to ensure that $Z_f[j]$ is a τ-function [13, 14]. Let us summarize the results as follows: If we assume stability of the vacuum, i.e. (50), the solution of the Schwinger-Dyson equation (55) satisfies

$$Z_f[j] \;=\; \tau[j]\,, \quad \text{where} \quad \begin{cases} \overline{W}{}^{(1)}_n\tau = 0 & \text{if } n \geq 1 \\ \overline{W}{}^{(2)\prime}_n\tau = 0 & \text{if } n \geq -1 \end{cases}\,, \tag{78}$$

i.e. $Z_f[j]$ is (according to general beliefs) a τ-function, as indicated by the notation in eq. (78).

3 Multicritical one-matrix models

Type $(2, 2m-1)$ conformal field theories coupled to quantum gravity are described by multicritical matrix models and we know the expressions for the functions $\Omega_1(\zeta_1)$, $\Omega_2(\zeta_1,\zeta_2)$, $F_N^{\mathrm{univ}(h)}(\zeta_1,\ldots,\zeta_N;\mu)$ and $f_N^{(h)}(\zeta_1,\ldots,\zeta_N;\mu)$, as well as the non-universal part $\lambda(\zeta)$. The transfer matrix has been derived in [15], while the string field theory expressed by $\Phi^\dagger(\zeta)$ and $\Psi(\eta)$ has been developed in [4]. We can repeat the calculations in the last section for the $(2, 2m-1)$ models and for all $m \geq 2$ it is now possible to write

$$\mathcal{H}^\star \;=\; -2\sqrt{G}\,\overline{W}{}^{(3)}_{-2} + Y\,, \tag{79}$$

where Y is a sum of terms which all contain some operators α_{2n} to the right and therefore annihilate $Z_f[j]$. It is possible to adjust the coefficients of the terms in Y without changing the Hamiltonian Schwinger-Dyson equations and as in pure gravity we can find a Y term such that $\mathcal{H}^\star$ in addition satisfies $\langle \mathrm{vac}|\mathcal{H}^\star = 0$. The string field and its current are related with the operators α_n through (71) and (72) with $k = 2m-3$. The Hamiltonian (79) is equivalent to that obtained in [4] up to a replacement of the first few of $\phi_1^\dagger$, $\phi_2^\dagger$, $\phi_3^\dagger$, … with $f_1^{(0)}(1;\mu)$, $f_1^{(0)}(2;\mu)$, $f_1^{(0)}(3;\mu)$, …. In the case of $G = 0$, the Hamiltonians in (79), in ref. [4], and in ref. [16], are all equivalent to each other up to some derivatives with respect to z in front of $\phi(z)$. We postpone the problem of understanding this difference to future studies.

If we define [17]

$$nx_n \;=\; \sqrt{G}nj_n + \frac{v_n}{\sqrt{G}}\sum_{k=0}^{m-1}\delta_{n,4k+1}\,, \tag{80}$$

v_1, v_5, v_9, … are determined by the Schwinger-Dyson equations and we get

$$Y \;=\; \frac{1}{2\sqrt{G}}\sum_{l=0}^{m-1}\sum_{k=\max(0,1-l)}^{m-1} v_{4l+1}v_{4k+1}\alpha_{4l+4k-2} + \theta_{m,3}\sum_{l=2}^{m-1}\sum_{k=1}^{2l-2} v_{4l+1}\alpha_{4l-2k-3}\alpha_{2k}\,. \tag{81}$$

This choice of v_n amounts to a specific choice of so-called conformal background [17].

4 The Ising model coupled to gravity

At the critical point the Ising model coupled to gravity describes a $(p,q) = (3,4)$ conformal field theory coupled to gravity. A new aspect appears compared to the situation above. Let us use the matrix model representation to describe the situation. The matrix model action will be

$$S(A_+,A_-) \;=\; N\,\mathrm{Tr}\Big(\frac{1}{2}A_+^2 + \frac{1}{2}A_-^2 - \kappa' A_+A_- - \frac{\kappa}{3}A_+^3 - \frac{\kappa}{3}A_-^3\Big)\,, \tag{82}$$

where A_+ and A_- are Hermitian $N\times N$ matrices which can be associated with "+" and "−" links. The matrix model will generate dynamical triangulations where the triangles have + or − spin and the A_+A_- term allows us in addition to glue together + and − triangles. The boundary configuration is characterized by a succession of

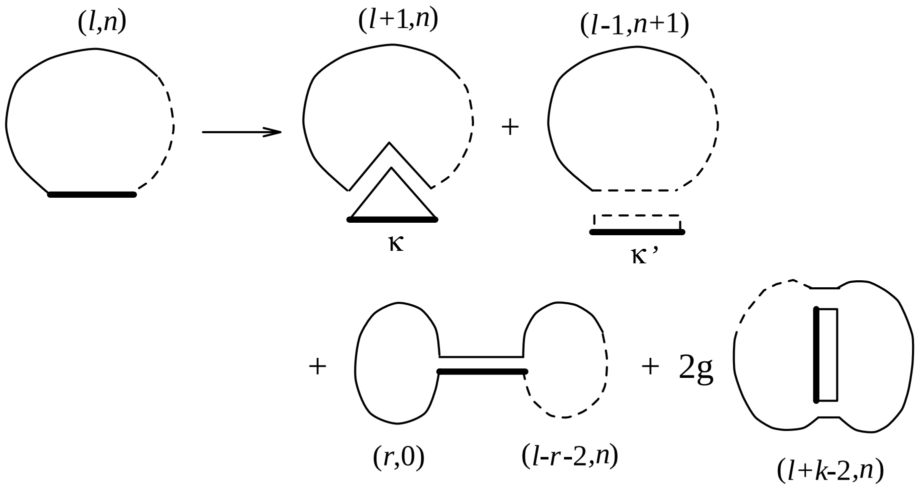

Figure 4: The deformation in the case of Ising spins which is almost equivalent to the one for pure gravity. The solid lines represent + spins on the boundary, the dashed lines − spins on the boundary, while the thick solid line represents the + spin which is being deformed. Note that we only consider special spin configurations on the boundary.

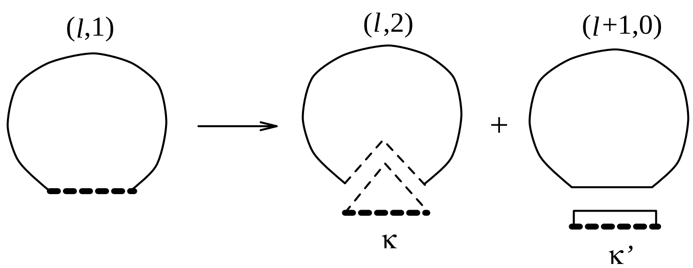

Figure 5: The deformation of a − spin (the thick dashed link), in the case where the boundary only has one − spin. The notation is as in Fig. 4.

+ and − links. It has not yet been possible to solve the discretized theory for arbitrary boundary spin configurations and we expect that most such boundary spin configurations are not important at the critical point. It *is* possible to solve the loop equations for the restricted set of boundary spin configurations where a boundary consisting of $n+l$ links has l neighboring + spins and n neighboring − spins [18]. Let us denote such a string state by $\Psi_n^\dagger(l)$. The deformation of $\Psi_n^\dagger(l)$ at the point of the boundary where the + and the − links meet is almost like the deformation of $\Psi^\dagger(l)$ in the case of pure gravity. The only difference occurs when the + spin at the boundary is glued to the rest of the triangulation by means of the interaction term A_+A_- or when we have a single − spin, or vice verse. We have illustrated these situations in Fig. 4 and Fig. 5. The two equations which replace the string deformation equation of pure gravity become:

$$\langle \delta\Psi_n^\dagger(l)\prod_i \Psi_0^\dagger(k_i)\rangle \;=\; \sum_{r=0}^{l-2}\Big(\langle \Psi_0^\dagger(r)\Psi_n^\dagger(l-r-2)\prod_i \Psi_0^\dagger(k_i)\rangle$$

$$
\begin{aligned}
&+ \sum_S \langle \Psi_0^\dagger(r) \prod_{i\in S} \Psi_0^\dagger(k_i)\rangle \langle \Psi_n^\dagger(l-r-2) \prod_{j\in\bar{S}} \Psi_0^\dagger(k_j)\rangle\Big) \\
&+ 2g\sum_j k_j \langle \Psi_n^\dagger(l+k_j-2) \prod_{i\neq j} \Psi_0^\dagger(k_i)\rangle \\
&+ \kappa\langle \Psi_n^\dagger(l+1) \prod_i \Psi_0^\dagger(k_i)\rangle \;+\; \kappa'\langle \Psi_{n+1}^\dagger(l-1) \prod_i \Psi_0^\dagger(k_i)\rangle \\
&- \langle \Psi_n^\dagger(l) \prod_i \Psi_0^\dagger(k_i)\rangle\,,
\end{aligned}
\tag{83}
$$

for $l \geq 1$ and $n \geq 0$, and

$$
\begin{aligned}
\langle \tilde{\delta}\Psi_1^\dagger(l) \prod_i \Psi_0^\dagger(k_i)\rangle \;=\; & \kappa\langle \Psi_2^\dagger(l) \prod_i \Psi_0^\dagger(k_i)\rangle \;+\; \kappa'\langle \Psi_0^\dagger(l+1) \prod_i \Psi_0^\dagger(k_i)\rangle \\
& - \langle \Psi_1^\dagger(l) \prod_i \Psi_0^\dagger(k_i)\rangle\,,
\end{aligned}
\tag{84}
$$

for $l \geq 0$. We here define $\Psi_0^\dagger(l=0)=1$. The notation is as in (25) except that κ' includes in addition the effect of a spin flip, as is clear from the discussion above and eq. (82).

Recursively using eqs. (83) and (84), it is possible to get a closed equation for $\Psi_0^\dagger(l)$. If we introduce a new deformation,

$$
\begin{aligned}
\langle \delta^{\mathrm{new}}\Psi_0^\dagger(l) \prod_i \Psi_0^\dagger(k_i)\rangle \;\stackrel{\mathrm{def}}{\equiv}\; & \sum_{r=1}^{l-1}\Big(\langle \delta\Psi_0^\dagger(r)\Psi_0^\dagger(l-r-1) \prod_i \Psi_0^\dagger(k_i)\rangle \\
&+ \sum_S \langle \delta\Psi_0^\dagger(r) \prod_{i\in S} \Psi_0^\dagger(k_i)\rangle \langle \Psi_0^\dagger(l-r-1) \prod_{j\in\bar{S}} \Psi_0^\dagger(k_j)\rangle\Big) \\
&+ 2g\sum_j k_j \langle \delta\Psi_0^\dagger(l+k_j-1) \prod_{i\neq j} \Psi_0^\dagger(k_i)\rangle \\
&+ \langle (\frac{\kappa'}{\kappa}\delta\Psi_0^\dagger(l) + \kappa\delta\Psi_0^\dagger(l+2) - \delta\Psi_0^\dagger(l+1)) \prod_i \Psi_0^\dagger(k_i)\rangle \\
&- \kappa'\langle (\delta\Psi_1^\dagger(l) - \frac{\kappa'}{\kappa}\tilde{\delta}\Psi_1^\dagger(l-1)) \prod_i \Psi_0^\dagger(k_i)\rangle\,,
\end{aligned}
\tag{85}
$$

then we get

$$
\langle \delta^{\mathrm{new}}\Psi_0^\dagger(l) \prod_i \Psi_0^\dagger(k_i)\rangle = \sum_{\substack{0\leq r,\ 0\leq s \\ r+s\leq l-3}} \langle \Psi_0^\dagger(r)\Psi_0^\dagger(s)\Psi_0^\dagger(l-r-s-3) \prod_i \Psi_0^\dagger(k_i)\rangle + \ldots\ldots \tag{86}
$$

In Appendix 1 we present the exact form of (the Laplace transformed of) $\delta^{\mathrm{new}}\Psi_0^\dagger(l)$. The important thing to note here is that the terms on the right hand side of eq. (86) only contain reference to $\Psi_0^\dagger(l)$, i.e. we can develop a *string field theory for + loops only*. Of course it would be desirable to have a complete string field theory which includes all possible spin loops, but it seems presently quite complicated to develop such theory.[2] Here we confine ourselves to the sector of the complete string field

[2] R. Nakayama has taken the first step in this direction by formulating a string field theory which involves + loops *and* − loops [19].

theory which only involves + loops. As shown above this is possible, but the price we pay is that the deformation $\delta^{\mathrm{new}}\Psi_0^\dagger(l)$ looses the simple relation to geodesic distance which was present in the case of pure gravity: moving one triangle "forward" all around the boundary should correspond to a step of one lattice unit, i.e. a step of length one in geodesic (lattice) units.

The two set of Schwinger-Dyson equations which determine the loop correlations have the form [18],

$$\lim_{T\to\infty}\langle\delta\Psi_n^\dagger(l)\prod_i\Psi_0^\dagger(k_i)\rangle \;=\; 0\,, \qquad \lim_{T\to\infty}\langle\tilde{\delta}\Psi_1^\dagger(l)\prod_i\Psi_0^\dagger(k_i)\rangle \;=\; 0\,. \tag{87}$$

In particular,

$$\lim_{T\to\infty}\langle\delta^{\mathrm{new}}\Psi_0^\dagger(l)\prod_i\Psi_0^\dagger(k_i)\rangle \;=\; 0\,, \tag{88}$$

will be reproduced in a SFT context if we find a Hamiltonian $\mathcal{H}$ such that

$$[\mathcal{H},\Psi_0^\dagger(l)] \;=\; -\,l\delta^{\mathrm{new}}\Psi_0^\dagger(l)\,. \tag{89}$$

We find after some tedious algebra along the lines discussed in the last section after taking the continuum limit

$$\begin{aligned}
\mathcal{H}^\star \;\;=\;\; & -\frac{1}{3}\sum_{a=b+c+d+9} aj_a\frac{\partial}{\partial j_b}\frac{\partial}{\partial j_c}\frac{\partial}{\partial j_d} \;-\; \frac{G}{2}\sum_{a+b=c+d+9} aj_a bj_b\frac{\partial}{\partial j_c}\frac{\partial}{\partial j_d} \\
& -\frac{G^2}{3}\sum_{a+b+c=d+9} aj_a bj_b cj_c\frac{\partial}{\partial j_d} \;-\; \frac{G}{6}\sum_{a=b+9}(ab+22)aj_a\frac{\partial}{\partial j_b} \\
& -v_7\sum_{a=b+c+2} aj_a\frac{\partial}{\partial j_b}\frac{\partial}{\partial j_c} \;-\; v_1\sum_{a=b+c+8} aj_a\frac{\partial}{\partial j_b}\frac{\partial}{\partial j_c} \\
& -Gv_7\sum_{a+b=c+2} aj_a bj_b\frac{\partial}{\partial j_c} \;-\; Gv_1\sum_{a+b=c+8} aj_a bj_b\frac{\partial}{\partial j_c} \\
& -v_7^2\sum_{a+5=b} aj_a\frac{\partial}{\partial j_b} \;-\; 2v_7v_1\sum_{a=b+1} aj_a\frac{\partial}{\partial j_b} \;-\; v_1^2\sum_{a=b+7} aj_a\frac{\partial}{\partial j_b} \\
& -2G^3\Big\{(j_1+\frac{v_1}{G})^3 j_6 \;+\; 5(j_1+\frac{v_1}{G})^2 j_2 j_5 \;+\; 6(j_1+\frac{v_1}{G})^2 j_3 j_4 \\
& \qquad +\; 8(j_1+\frac{v_1}{G})j_2^2 j_4 \;+\; 9(j_1+\frac{v_1}{G})j_2 j_3^2 \;+\; 4j_2^3 j_3\Big\} \\
& -\frac{4G^2}{3}\Big\{14(j_1+\frac{v_1}{G})j_8 \;+\; 14j_2(j_7+\frac{v_7}{7G}) \;+\; 9j_3j_6 \;+\; 5j_4j_5\Big\} \\
& -\frac{v_7^2}{G}(\frac{\partial}{\partial j_1}\frac{\partial}{\partial j_4}+\frac{\partial}{\partial j_2}\frac{\partial}{\partial j_3}) \;-\; \frac{v_7^3}{3G}\frac{\partial}{\partial j_{12}} \;-\; \frac{v_7^2 v_1}{G}\frac{\partial}{\partial j_6} \;+\; 9GY\,.
\end{aligned} \tag{90}$$

Here Y denotes a sum of terms which all contain some operators $\frac{\partial}{\partial j_{3n}}$ in the same way as Y in the last section denoted a sum of terms which all contains some operators $\frac{\partial}{\partial j_{2n}}$. They are singled out because

$$\frac{\partial}{\partial j_{3n}}Z_f[j]=0, \tag{91}$$

the analogy of (75) for $p = 3$ in the (p, q) system. The string field and its current are related with the operators α_n through (71) and (72) with $k = 1$.

It is now possible to proceed as in the last section and introduce

$$nx_n \;=\; \sqrt{G} n j_n \,+\, \frac{v_1}{\sqrt{G}}\delta_{n,1} \,+\, \frac{v_7}{\sqrt{G}}\delta_{n,7}\,, \quad \frac{\partial}{\partial x_n} \;=\; \frac{1}{\sqrt{G}}\frac{\partial}{\partial j_n}\,, \tag{92}$$

where the values of v_1 and v_7 are $-\mu/3$ and 1, respectively (but it is not important for the arguments to follow). We then find that (90) is rewritten as

$$\mathcal{H}^\star \;=\; -\,9G\,\overline{W}{}^{(4)}_{-3} \,+\, Y\,. \tag{93}$$

Using $\frac{\partial}{\partial j_{3n}} Z_f[j] = 0$, the constraint $\mathcal{H}^\star Z_f[j] = 0$ becomes

$$\begin{aligned}\overline{W}{}^{(4)}_{-3}\, Z_f[j] \;&=\; \Big(\sum_{n=1}^{\infty} \alpha_{-3n}\overline{W}{}^{(3)}_{n-3} \,-\, \frac{1}{3}\sum_{n=2}^{\infty}\sum_{a+b=-3n} :\alpha_a^{(0)}\alpha_b^{(0)} - \alpha_a^{(1)}\alpha_b^{(2)}: \overline{W}{}^{(2)}_{n-3}\Big) Z_f[j] \\ &=\; 0\,. \end{aligned} \tag{94}$$

In this way we finally arrive at the following constraints:

$$\overline{W}{}^{(3)}_{n-3}\, Z_f[j] \;=\; \overline{W}{}^{(2)}_{n-2}\, Z_f[j] \;=\; \overline{W}{}^{(1)}_{n-1} Z_f[j] \;=\; 0 \qquad \text{for } n \geq 1. \tag{95}$$

In order to satisfy the vacuum condition (50), we may try to determine Y in (93) as was done in the pure gravity case. However, *in Ising case it is impossible to find a Y such that the vacuum condition (50) is satisfied*, since we see from the last line in eq. (90) that $\overline{W}{}^{(4)}_{-3}$ contains a term

$$-\,\frac{v_7^2}{G}\,\frac{\partial}{\partial j_1}\frac{\partial}{\partial j_4}, \tag{96}$$

which has no reference to $\partial/\partial j_{3n}$. But in the case of the Ising model we have a larger freedom to add terms while still satisfying the Schwinger-Dyson equations since $Z_f[j]$ is not only annihilated by $\overline{W}{}^{(1)}_n$ operators but also by $\overline{W}{}^{(2)}_n$ operators. We can use these to modify the Hamiltonian without modifying the Schwinger-Dyson equations. Potential candidate terms which have the right dimension are properly normal ordered terms like

$$\int \frac{dz}{2\pi i} : (\overline{W}{}^{(2)}(z))^2 : \tag{97}$$

and similar products which involve $\overline{W}{}^{(2)}(z)(\overline{W}{}^{(1)}(z))^2$, etc. However, these last terms all contain $\overline{W}{}^{(1)}(z)$ and they can be absorbed in the definition of Y. In this way the coefficient of the term (97) is uniquely fixed by the requirement that it should cancel the term (96), and we get

$$\mathcal{H}^\star \;=\; -\,9GX \,+\, Y\,, \quad X \;=\; \overline{W}{}^{(4)}_{-3} \,-\, \sum_{n=2}^{\infty} \overline{W}{}^{(2)}_{-n}\overline{W}{}^{(2)}_{n-3}\,, \tag{98}$$

where Y is still undetermined. After the elimination of (96), the situation for $\mathcal{H}^\star$ in (98) is the same as in the $p=2$ case and the term Y is now uniquely determined by requiring that the vacuum condition (50) is satisfied. We find

$$Y \;=\; \frac{v_7^3}{3G^{1/2}}\alpha_{12} \,+\, \frac{v_7^2 v_1}{G^{1/2}}\alpha_6 \,+\, v_7^2\alpha_2\alpha_3\,. \tag{99}$$

Therefore, we obtain

$$\mathcal{H}^\star \;=\; -\,9G\left(\overline{W}{}^{(4)}_{-3} - \sum_{n=2}^{\infty}\overline{W}{}^{(2)}_{-n}\overline{W}{}^{(2)}_{n-3}\right) \,+\, \frac{v_7^3}{3G^{1/2}}\alpha_{12} \,+\, \frac{v_7^2 v_1}{G^{1/2}}\alpha_6 \,+\, v_7^2\alpha_2\alpha_3\,. \tag{100}$$

The explicit form of $\mathcal{H}$ corresponding to $\mathcal{H}^\star$ is given in the Appendix 2.

Since

$$\begin{aligned}\overline{W}{}^{(4)}_{-3} - \sum_{n=2}^{\infty}\overline{W}{}^{(2)}_{-n}\overline{W}{}^{(2)}_{n-3} \;=\;& -\frac{1}{216}\sum_{a+b+c+d=-3} :\overline{W}{}^{(1)}_{a}\overline{W}{}^{(1)}_{b}\overline{W}{}^{(1)}_{c}\overline{W}{}^{(1)}_{d}: \\ &+\frac{1}{6}\sum_{a+b+c=-3} :\overline{W}{}^{(1)}_{a}\overline{W}{}^{(1)}_{b}:\overline{W}{}^{(2)\prime}_{c} \,+\, \sum_{a+b=-3}\overline{W}{}^{(1)}_{a}\overline{W}{}^{(3)\prime}_{b}\,,\end{aligned} \tag{101}$$

we conclude that there exists a Hamiltonian with a stable vacuum such that the Hamiltonian Schwinger-Dyson equations has a solution $Z_f[j]$ which is characterized by

$$Z_f[j] \;=\; \tau[j]\,, \qquad \begin{cases}\overline{W}{}^{(1)}_{n}\tau = 0 & \text{if } n \geq 1 \\ \overline{W}{}^{(2)\prime}_{n}\tau = 0 & \text{if } n \geq -1 \\ \overline{W}{}^{(3)\prime}_{n}\tau = 0 & \text{if } n \geq -2\end{cases}\,. \tag{102}$$

These conditions are believed to imply that $Z_f[j]$ is a τ-function, as we have indicated by the notation used in eq. (102).

5 Gravity coupled to (p,q) conformal fields

The procedures outlined above can be generalized to any (p,q) model coupled to gravity. The algebraic details are rather tedious so we will confine ourselves to state the results.

We define as before

$$nx_n \;=\; \sqrt{G}nj_n + \frac{v_n}{\sqrt{G}}\,, \quad \frac{\partial}{\partial x_n} \;=\; \frac{1}{\sqrt{G}}\frac{\partial}{\partial j_n}\,, \tag{103}$$

and obtain the Hamiltonian

$$\mathcal{H}^\star \;=\; -\,p^{(p-1)}G^{(p-1)/2}X \,+\, Y\,. \tag{104}$$

Y denotes a sum of terms which all contain some operators $\frac{\partial}{\partial j_{pn}}$ and will be determined by the vacuum condition (50). The problem of a stable vacuum is as in the Ising model, only more severe. Again we find that it can be repaired by modifying the Hamiltonian with terms which do not interfere with the requirement that $Z_f[j]$

is a τ-function. After adding such terms we can find a Y such that $\mathcal{H}^\star$ has a stable vacuum.

In the case of $(p,q)=(4,5)$ we find:

$$
\begin{aligned}
X &= \overline{W}^{(5)}_{-4} - \sum_{n=2}^{\infty} \overline{W}^{(2)}_{-n}\overline{W}^{(3)}_{n-4} - \sum_{n=3}^{\infty} \overline{W}^{(3)}_{-n}\overline{W}^{(2)}_{n-4}\,, \\
Y &= \frac{v_9^4}{4G^{1/2}}\alpha_{20} + \frac{v_9^3 v_1}{G^{1/2}}\alpha_{12} + \frac{3v_9^2 v_1^2}{2G^{1/2}}\alpha_4 + v_9^3(\alpha_3\alpha_8 + \alpha_7\alpha_4)\,, \qquad (105)
\end{aligned}
$$

and we can alternative write X as

$$
\begin{aligned}
X = & -\frac{7}{4^4 15} \sum_{a+b+c+d+e=-4} :\overline{W}^{(1)}_a\overline{W}^{(1)}_b\overline{W}^{(1)}_c\overline{W}^{(1)}_d\overline{W}^{(1)}_e: \\
& -\frac{1}{48}\sum_{a+b+c+d=-4} :\overline{W}^{(1)}_a\overline{W}^{(1)}_b\overline{W}^{(1)}_c:\overline{W}^{(2)\prime}_d \\
& +\frac{1}{4}\sum_{a+b+c=-4} :\overline{W}^{(1)}_a\overline{W}^{(1)}_b:\overline{W}^{(3)\prime}_c + \sum_{a+b=-4}\overline{W}^{(1)}_a\overline{W}^{(4)\prime}_b\,, \qquad (106)
\end{aligned}
$$

which makes it easy to understand that $Z_f[j]=\tau[j]$. For the $(5,6)$ model we get

$$
\begin{aligned}
X &= \overline{W}^{(6)}_{-5} - \sum_{n=2}^{\infty}\overline{W}^{(2)}_{-n}\overline{W}^{(4)}_{n-5} - \sum_{n=3}^{\infty}\overline{W}^{(3)}_{-n}\overline{W}^{(3)}_{n-5} - \sum_{n=4}^{\infty}\overline{W}^{(4)}_{-n}\overline{W}^{(2)}_{n-5} \qquad (107) \\
&\quad + \frac{1}{2}\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} {}^\circ_\circ\overline{W}^{(2)}_{-n}\overline{W}^{(2)}_{-m}\overline{W}^{(2)}_{n+m-5}{}^\circ_\circ + \frac{1}{2}\sum_{n=-1}^{\infty}\sum_{m=-1}^{\infty} {}^\circ_\circ\overline{W}^{(2)}_{-n-m-5}\overline{W}^{(2)}_{n}\overline{W}^{(2)}_{m}{}^\circ_\circ\,, \\
Y &= \frac{v_{11}^5}{5G^{1/2}}\alpha_{30} + \frac{v_{11}^4 v_1}{G^{1/2}}\alpha_{20} + \frac{2v_{11}^3 v_1^2}{G^{1/2}}\alpha_{10} + v_{11}^4(\alpha_4\alpha_{15}+\alpha_9\alpha_{10}+\alpha_{14}\alpha_5) \\
&\quad + 4v_{11}^3 v_1\alpha_4\alpha_5 - G^{1/2}v_{11}^3\alpha_1\alpha_2\alpha_5\,. \qquad (108)
\end{aligned}
$$

The string field and its current are related with the operators α_n through (71) and (72) with $k=1$ for any $(m,m+1)$ model.

It is seen that an interesting algebraic structure is present since we can write

$$
\begin{aligned}
X &= -\oint\frac{dz}{2\pi i}\oint\frac{ds}{2\pi i}\, s^{-p-2}\,{}^\circ_\circ\exp[-\overline{W}(z,s)]{}^\circ_\circ \\
&= \overline{W}^{(p+1)}_{-p} - \frac{1}{2}\sum_{k=2}^{p-1}\sum_{n\in Z} {}^\circ_\circ\overline{W}^{(k)}_{-n}\overline{W}^{(p-k+1)}_{n-p}{}^\circ_\circ \\
&\quad + \frac{1}{3!}\sum_{\substack{k\geq 2,\ l\geq 2\\ k+l+1\leq p}}\sum_{n,m\in Z} {}^\circ_\circ\overline{W}^{(k)}_{-n}\overline{W}^{(l)}_{-m}\overline{W}^{(p-k-l+1)}_{n+m-p}{}^\circ_\circ + \ldots\ldots \qquad (109)
\end{aligned}
$$

with

$$
\overline{W}(z,s) = \sum_{k=2}^{\infty}\overline{W}^{(k)}(z)s^k\,, \qquad \overline{W}^{(k)}(z) = \sum_{n\in Z}\overline{W}^{(k)}_n z^{-n-k}\,. \qquad (110)
$$

With these definitions the Hamiltonian (104) is valid for any (p,q) model. It should be noted that one in eq. (109) encounters a problem defining the normal ordering

of the product of more than three $\overline{W}$ operators. We also note that one could in principle define the following more general X operator,

$$\circ\atop\circ \exp[-\overline{W}_{[r]}(z,s)]{\circ\atop\circ} \;=\; 1 \,-\, \sum_{k=r}^{\infty}\sum_{n\in Z} X^{(k)}_{[r]n} z^{-n-k} s^k \tag{111}$$

with

$$\overline{W}_{[r]}(z,s) \;=\; \sum_{k=r}^{\infty} \overline{W}^{(k)}(z) s^k \,, \qquad \overline{W}^{(k)}(z) \;=\; \sum_{n\in Z} \overline{W}^{(k)}_n z^{-n-k} \,. \tag{112}$$

It seems that only $X = X^{(p+1)}_{[2]-p}$ plays a role for (p,q) models coupled to quantum gravity. We further make the surprising observation that $X^{(p+1)}_{[1]-p} = 0$ for $p = 2$ and $p = 3$ and conjecture that it is true for all p.

In ref. [17], the authors give the explicit form of the disk amplitudes for any (p,q) model. Then we have

$$F^{\mathrm{univ}(0)}_1(\zeta;\mu) \;=\; \Omega_1(\zeta) \,+\, \mathrm{const.}\mu^{1-\gamma}\zeta^{\gamma-1} \,+\, \ldots\ldots \,, \tag{113}$$

where $\Omega_1(\zeta)$ is a polynomial with respect to μ and has the form, $\Omega_1(\zeta) = \mathrm{const.}\zeta^{1-\gamma} + O(\mu)$. Here γ is the string susceptibility which has the value $\gamma = 1 - q/p$ for $p < q$ or $\gamma = 1 - p/q$ for $p > q$. Since $v(z)$ corresponds to $\Omega_1(\zeta)$, $v(z)$ is also a polynomial with respect to μ, i.e. we get

$$v(z) \;=\; \sum_{0\le i<h/(2p)} v_{h-2ip}\, z^{h-2ip-1} \,, \tag{114}$$

from $\dim\mu = \dim\zeta^2$ and $\zeta = z^p$, where $v_h = +1$ or -1. By using (10) one can compare the leading term of $\Omega_1(\zeta)$ in (113) with that of $v(z)$ in (72). Then we find $h = p+q+k-1$ if $p < q$. On the other hand, h becomes non-integer if $p > q$, which means that we cannot express the Hamiltonian $\mathcal{H}$ by the string fields.

We here require that the Hamiltonian for disk amplitude, $\mathcal{H}_{\mathrm{disk}} = \mathcal{H}|_{G=0}$, has a tadpole term. This requirement is quite natural because a tadpole is necessary in order to make a disk topology. From (104), (109), (63), (58), (59), (57), and (68), the above requirement leads to the fact that one of v_1, v_2, ..., v_{p-1} is non-zero. In order that this fact is consistent with (114), we reach the following most plausible conjecture for the general (p,q)-SFT with $p < q$: $h = p[q/p] + q$, which leads to $k = p([q/p]-1)+1$. This conjecture is satisfied in $(2,3)$-SFT, $(2,2m-1)$-SFT's and $(m,m+1)$-SFT's.

6 String Field Theory for One-string Propagation

In [1] it was shown that important variables in quantum gravity which refer to geodesic distance are conveniently obtained by using the transfer matrix,

$$G(n,m;T) \;=\; \langle\mathrm{vac}|\phi_m e^{-T\overline{\mathcal{H}}}\phi^\dagger_n|\mathrm{vac}\rangle \,, \tag{115}$$

where $\overline{\mathcal{H}}$ is the Hamiltonian, which expresses the propagation of one-string state,

$$\overline{\mathcal{H}} \;=\; \sum_{l=1}^{\infty} \phi_l^\dagger \left[\frac{\delta}{\delta \phi_l^\dagger} \mathcal{H} \right]_{\phi_l^\dagger \to f_l, G \to 0} , \tag{116}$$

where $f_l \equiv f_1^{(0)}(l;\mu)$ is the disk amplitude introduced earlier.

In pure gravity, we find

$$\overline{\mathcal{H}} \;=\; -\sum_{l=1}^{\infty} \phi_{l+1}^\dagger l \phi_l \,+\, \frac{3}{8}\mu \sum_{l=4}^{\infty} \phi_{l-3}^\dagger l \phi_l \,-\, \sum_{l=6}^{\infty} \sum_{k=1}^{l-5} f_{l-k-4} \phi_k^\dagger l \phi_l \,. \tag{117}$$

By using $\phi^\dagger(z)$ and $\phi(z)$, eq. (117) becomes

$$\overline{\mathcal{H}} \;=\; -\oint \frac{dz}{2\pi i} \frac{1}{z^2} \Big\{ \Big(-\frac{3}{8}\mu + z^4 + f(z) \Big) \phi^\dagger(z) \frac{\partial}{\partial z} \Big(z \phi(z) \Big) \Big\} , \tag{118}$$

where $f(z) = \sum_{n=1}^{\infty} f_n z^{-n-1}$. In the general case $(p,q) = (2, 2m+1)$, we find

$$\overline{\mathcal{H}} \;=\; -\oint \frac{dz}{2\pi i} \frac{1}{z^2} \Big\{ \Big(v(z) + f(z) \Big) \phi^\dagger(z) \frac{\partial}{\partial z} \Big(z \phi(z) \Big) \Big\} . \tag{119}$$

Using (115) we obtain the following differential equation for the transfer matrix,

$$\frac{\partial}{\partial T} G(x,y;T) \;=\; -\frac{\partial}{\partial x} \Big\{ \frac{1}{x^2} \Big(v(x) + f(x) \Big) G(x,y;T) \Big\}^{(-)} , \tag{120}$$

where $[x^n]^{(-)}$ means x^n if $n < 0$ and zero if $n \geq 0$. In pure gravity the differential equation (120) is the same equation that was obtained in [1]. On the other hand, in the multicritical model the eq. (120) is slightly different from that in [15]. We postpone the problem of understanding this difference to future studies.

In the gravity coupled to Ising matter, the Hamiltonian $\overline{\mathcal{H}}$ has the form

$$\begin{aligned} \overline{\mathcal{H}} \;=\; & -\sum_{a=b+c+d+9} f_d f_c \phi_b^\dagger a \phi_a + \sum_{n=4}^{\infty} \sum_{\substack{c+d=3n-9 \\ a=b+3n}} f_d f_b \phi_c^\dagger a \phi_a + \frac{1}{2} \sum_{n=4}^{\infty} \sum_{\substack{c+d=3n-9 \\ a=b+3n}} f_d f_c \phi_b^\dagger a \phi_a \\ & - v_7 \Big(2 \sum_{a=b+c+2} f_c \phi_b^\dagger a \phi_a - \sum_{n=1}^{\infty} \sum_{a=b+3n} f_{3n-2} \phi_b^\dagger a \phi_a - \sum_{n=1}^{\infty} \sum_{a=b+3n} f_b \phi_{3n-2}^\dagger a \phi_a \\ & \qquad - \sum_{n=1}^{\infty} \sum_{a+b=3n} f_b \phi_a^\dagger (3n+2) \phi_{3n+2} \Big) \\ & - v_1 \Big(2 \sum_{a=b+c+8} f_c \phi_b^\dagger a \phi_a - \sum_{n=3}^{\infty} \sum_{a=b+3n} f_{3n-8} \phi_b^\dagger a \phi_a - \sum_{n=3}^{\infty} \sum_{a=b+3n} f_b \phi_{3n-8}^\dagger a \phi_a \\ & \qquad - \sum_{n=1}^{\infty} \sum_{a+b=3n} f_b \phi_a^\dagger (3n+8) \phi_{3n+8} \Big) \\ & - v_7^2 \Big(\sum_{a+5=b} \phi_b^\dagger a \phi_a - \sum_{n=1}^{\infty} \phi_{3n+4}^\dagger (3n-1) \phi_{3n-1} \Big) \\ & - 2 v_7 v_1 \Big(\sum_{a=b+1} \phi_b^\dagger a \phi_a - \sum_{n=1}^{\infty} \phi_{3n-2}^\dagger (3n-1) \phi_{3n-1} \Big) \\ & - v_1^2 \Big(\sum_{a=b+7} \phi_b^\dagger a \phi_a - \sum_{n=1}^{\infty} \phi_{3n-2}^\dagger (3n+5) \phi_{3n+5} \Big) . \end{aligned} \tag{121}$$

In this case we cannot express (121) in a simple way in terms of $\phi^\dagger(z)$ and $\phi(z)$. Rather, the natural variables seem to be $\phi^{[i]\dagger}(z)$ and $\phi^{[i]}(z)$. Using these variables (121) can be written as by

$$
\begin{aligned}
\overline{\mathcal{H}} \;=\; \oint \frac{dz}{2\pi i}\frac{1}{z^6}\Bigg[& \Big(\frac{1}{2}(f^{[0]})^2 - f^{[1]}(v^{[1]}+f^{[2]})\Big)\phi^{[0]\dagger}\frac{\partial}{\partial z}(z\phi^{[0]}) \\
& -\frac{1}{2}(f^{[0]})^2\Big(\phi^{[1]\dagger}\frac{\partial}{\partial z}(z\phi^{[1]}) + \phi^{[2]\dagger}\frac{\partial}{\partial z}(z\phi^{[2]})\Big) \\
& -\Big(f^{[0]}(v^{[1]}+f^{[2]}) + (f^{[1]})^2\Big)\Big(\phi^{[1]\dagger}\frac{\partial}{\partial z}(z\phi^{[0]}) + \phi^{[0]\dagger}\frac{\partial}{\partial z}(z\phi^{[2]})\Big) \\
& -\Big(f^{[0]}f^{[1]} + (v^{[1]}+f^{[2]})^2\Big)\Big(\phi^{[2]\dagger}\frac{\partial}{\partial z}(z\phi^{[0]}) + \phi^{[0]\dagger}\frac{\partial}{\partial z}(z\phi^{[1]})\Big) \\
& -2f^{[0]}f^{[1]}\phi^{[1]\dagger}\frac{\partial}{\partial z}(z\phi^{[2]}) - 2f^{[0]}(v^{[1]}+f^{[2]})\phi^{[2]\dagger}\frac{\partial}{\partial z}(z\phi^{[1]})\Bigg].
\end{aligned} \tag{122}
$$

In the general case $(p,q) = (3,q)$, we find

$$
\overline{\mathcal{H}} \;=\; -\oint \frac{dz}{2\pi i}\frac{1}{z^6}\sum_{i=0}^{2}\sum_{j=0}^{2} M^{[i,j]}\phi^{[j]\dagger}\frac{\partial}{\partial z}(z\phi^{[i]})\,, \tag{123}
$$

where

$$
\begin{aligned}
M^{[0,0]} &= -\frac{1}{2}(v^{[0]}+f^{[0]})^2 + (v^{[2]}+f^{[1]})(v^{[1]}+f^{[2]})\,, \\
M^{[1,1]} &= M^{[2,2]} \;=\; \frac{1}{2}(v^{[0]}+f^{[0]})^2\,, \\
M^{[0,1]} &= M^{[2,0]} \;=\; (v^{[0]}+f^{[0]})(v^{[1]}+f^{[2]}) + (v^{[2]}+f^{[1]})^2\,, \\
M^{[0,2]} &= M^{[1,0]} \;=\; (v^{[0]}+f^{[0]})(v^{[2]}+f^{[1]}) + (v^{[1]}+f^{[2]})^2\,, \\
M^{[1,2]} &= 2(v^{[0]}+f^{[0]})(v^{[1]}+f^{[2]})\,, \\
M^{[2,1]} &= 2(v^{[0]}+f^{[0]})(v^{[2]}+f^{[1]})\,.
\end{aligned} \tag{124}
$$

The transfer matrix satisfies the differential equations,

$$
\frac{\partial}{\partial T}G^{[i,j]}(x,y;T) \;=\; -\frac{\partial}{\partial x}\Big\{\frac{1}{x^6}\sum_{k=0}^{2}M^{[i,k]}(x)G^{[k,j]}(x,y;T)\Big\}^{(-)}\,. \tag{125}
$$

7 Conclusion

Starting from first principles, i.e. from models regularized by the use of dynamical triangulations, we have shown that it is possible to define a string field Hamiltonian with the following properties: the vacuum is stable and the solution $Z_f[j]$ of the Hamiltonian Schwinger-Dyson equations is a τ-function of the kind expected from matrix model considerations.

How unique is this Hamiltonian? Here we have in the pure gravity case as well as the Ising model case relied on an explicit construction of the string loop deformation,

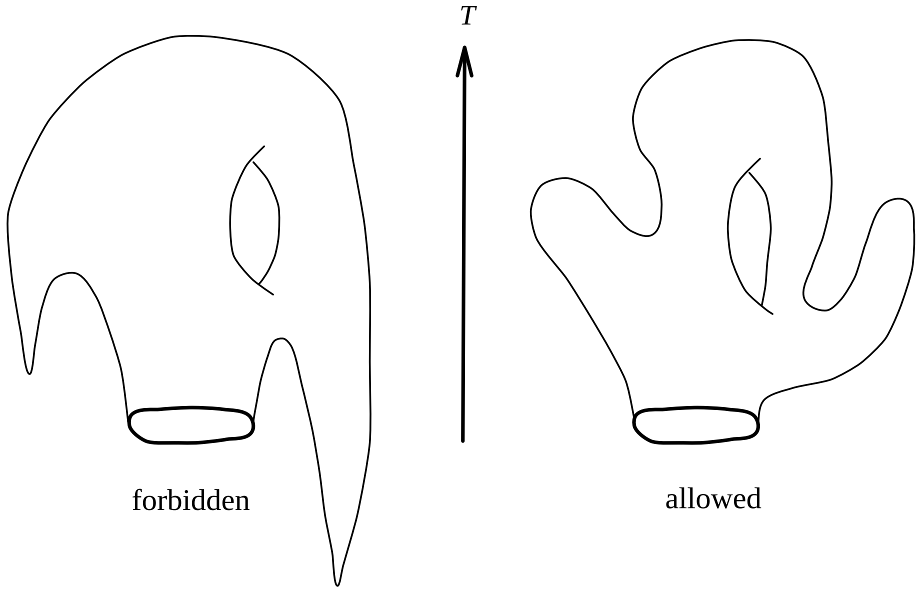

Figure 6: A forbidden configuration and an allowed configuration for a given choice of proper time slicing indicated by the arrow.

taken from the formalism of dynamical triangulations. Within this formalism, the Hamiltonian was unique up to terms which annihilated the generating functional $Z_f[j]$. This ambiguity was removed by the requirement of a stable vacuum. We can formulate the situation as follows: a given choice of string loop deformation corresponds to a specific choice of proper time slicing of our Euclidean space-time and the stability of the vacuum rules out the situation that non-trivial physics can be created out of the vacuum. We have shown such a forbidden situation in Fig. 6.

As we have seen the Hamiltonian is also (almost) uniquely determined in the case of the mth multicritical one matrix models since the factor $k = 2m - 3$. However, our transfer matrix is not completely identical to that of derived in [15]. This is potentially a good thing since the transfer matrix derived in [15] has a number of undesirable features for $m > 2$. We hope to analyze the problem further in the future. It is most likely intimately connected with the relation between the proper time and the geodesic distance. It is clear that the geodesic distance can be defined and used as a proper time parameter in the context of dynamical triangulations. In the case of pure two-dimensional gravity this choice coincides with the proper time defined by the transfer matrix via the fundamental deformation $\delta\Psi^\dagger(l)$ as discussed, and the geometrical reason for this is clear. In other cases the relation is less clear, and for instance in the Ising model we saw that the basic deformation $\delta^{\rm new}\Psi_0^\dagger(l)$ had already lost its simple geometric interpretation. At least the precise relation between the dimension of proper time and the dimension of geodesic distance should be understood. We do not presently know the internal Hausdorff dimension of our (p,q) quantum universes and consequently not the dimension of geodesic distance, but we can relate the dimension of $\mathcal{H}$ and therefore that of the proper time T, $\dim T = -\dim \mathcal{H}$ to the choice of conformal background v_n. The length of the boundary L is connected to the variable ζ via Laplace transformations like (8),

i.e. $\dim\zeta = -\dim L$. For a (p,q) model the variable $z = \zeta^{1/p}$ and $\dim X = \dim \overline{W}^{(p+1)}_{-p} = \dim z^{-p^2}$, see sec. 2.3, sec. 5 and appendix 3. From (103) and (114) we conclude that $\dim\sqrt{G} = \dim z^h$, and we finally get

$$\dim T = -\dim\mathcal{H} = -\dim\Big(G^{\frac{p-1}{2}}\ X\Big) = \dim L^{\frac{(p-1)(h-p)}{p}-1}. \tag{126}$$

For pure gravity $h=5$, see (73), then eq. (126) indeed reproduces $\dim T = \dim L^{1/2}$.

We have shown by explicit construction how the $W^{(2)}$ and $W^{(3)}$ constraints appear in the context of the Hamiltonian formalism in pure gravity and argued that they generalize to any (p,q) model coupled to two-dimensional quantum gravity. It is a pleasant surprise that the Schwinger-Dyson equations determined the form of $\mathcal{H}^\star$ to be

$$\mathcal{H}^\star \;=\; -p^{(p-1)}G^{(p-1)/2}X + Y, \qquad X \;=\; \overline{W}^{(p+1)}_{-p} - \ldots\ldots, \tag{127}$$

but even more surprising that the vacuum condition seems to organize the structure of X as

$$X \;=\; -\oint\frac{dz}{2\pi i}\oint\frac{ds}{2\pi i}\, s^{-p-2}\,{}^\circ_\circ \exp[-\overline{W}(z,s)]^\circ_\circ\,. \tag{128}$$

Nothing is known about this algebraic structure which seems to organize the vacuum of two-dimensional quantum gravity coupled to matter.

We have relied on the Schwinger-Dyson equations as derived from the matrix models as a guiding principle for deriving the form of the Hamiltonian. It is an interesting question to which extension it is possible to derive the W-constraints entirely within a Hamiltonian context. For instance, is it possible to derive $\overline{W}^{(1)}_n Z_f[j] = 0$ from $\mathcal{H}^\star Z = 0$?

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Appendix 1

$$\begin{aligned}
\delta^{\rm new}\Psi_0^\dagger(x) = \Big[\, & x^3\Big(\Psi_0^\dagger(x)\Big)^3 \;+\; \Big(\frac{\kappa' x^2}{\kappa} - 2x + 2\kappa\Big)\Big(\Psi_0^\dagger(x)\Big)^2 \\
& +\; \Big(\frac{\kappa^2}{x^3} - \frac{2\kappa}{x^2} - \frac{\kappa'+1}{x} + \frac{\kappa'^3}{\kappa} - \frac{\kappa'}{\kappa} - \kappa + x\Big)\Psi_0^\dagger(x) \;-\; \kappa\frac{\partial\Psi_0^\dagger(0)}{\partial x}x\Psi_0^\dagger(x) \\
& +\; 2g\Big\{x\Big[x^2\Psi_0^\dagger(x)\big(-x\frac{\partial}{\partial x}\Psi_0(\frac{1}{x})\big)\Big]^{(+)}\Psi_0^\dagger(x)
\end{aligned}$$

$$
\begin{aligned}
&+\ x^3\Big(\Psi_0^\dagger(x)\Big)^2(-x\frac{\partial}{\partial x}\Psi_0(\frac{1}{x}))\Big\}\\
&+\ 2g(\frac{\kappa' x^2}{\kappa}-2x+2\kappa)\Psi_0^\dagger(x)(-x\frac{\partial}{\partial x}\Psi_0(\frac{1}{x}))\\
&+\ 4g^2x^3\Psi_0^\dagger(x)(-x\frac{\partial}{\partial x}\Psi_0(\frac{1}{x}))^2\Big]^{(+)}.
\end{aligned} \tag{129}
$$

Appendix 2

The explicit form of the Hamiltonian (100) is

$$
\begin{aligned}
\mathcal{H} \;=\; & -\frac{1}{3}\sum_{a=b+c+d+9}\phi_d^\dagger\phi_c^\dagger\phi_b^\dagger a\phi_a \;+\; \frac{1}{2}\sum_{n=4}^{\infty}\sum_{\substack{c+d=3n-9\\ a=b+3n}}\phi_d^\dagger\phi_c^\dagger\phi_b^\dagger a\phi_a\\
& -\frac{G}{2}\sum_{a+b=c+d+9}\phi_d^\dagger\phi_c^\dagger b\phi_b a\phi_a \;+\; G\sum_{n=2}^{\infty}\sum_{\substack{b+3n-9=d\\ a=c+3n}}\phi_d^\dagger\phi_c^\dagger b\phi_b a\phi_a\\
& +\frac{G}{4}\sum_{n=4}^{\infty}\sum_{\substack{c+d=3n-9\\ a+b=3n}}\phi_d^\dagger\phi_c^\dagger b\phi_b a\phi_a - \frac{G^2}{3}\sum_{a+b+c=d+9}\phi_d^\dagger c\phi_c b\phi_b a\phi_a\\
& +\frac{G^2}{2}\sum_{n=1}^{\infty}\sum_{\substack{c+3n-9=d\\ a+b=3n}}\phi_d^\dagger c\phi_c b\phi_b a\phi_a\\
& -v_7\Big(\sum_{a=b+c+2}\phi_c^\dagger\phi_b^\dagger a\phi_a - \sum_{n=1}^{\infty}\sum_{a=b+3n}\phi_{3n-2}^\dagger\phi_b^\dagger a\phi_a\\
& \qquad -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{a+b=3n}\phi_b^\dagger\phi_a^\dagger(3n+2)\phi_{3n+2}\Big)\\
& -v_1\Big(\sum_{a=b+c+8}\phi_c^\dagger\phi_b^\dagger a\phi_a - \sum_{n=3}^{\infty}\sum_{a=b+3n}\phi_{3n-8}^\dagger\phi_b^\dagger a\phi_a\\
& \qquad -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{a+b=3n}\phi_b^\dagger\phi_a^\dagger(3n+8)\phi_{3n+8}\Big)\\
& -Gv_7\Big(\sum_{a+b=c+2}\phi_c^\dagger b\phi_b a\phi_a - \frac{1}{2}\sum_{n=1}^{\infty}\sum_{a+b=3n}\phi_{3n-2}^\dagger b\phi_b a\phi_a\\
& \qquad -\sum_{n=0}^{\infty}\sum_{a+3n=b}\phi_b^\dagger a\phi_a(3n+2)\phi_{3n+2}\Big)\\
& -Gv_1\Big(\sum_{a+b=c+8}\phi_c^\dagger b\phi_b a\phi_a - \frac{1}{2}\sum_{n=3}^{\infty}\sum_{a+b=3n}\phi_{3n-8}^\dagger b\phi_b a\phi_a\\
& \qquad -\sum_{n=-2}^{\infty}\sum_{a+3n=b}\phi_b^\dagger a\phi_a(3n+8)\phi_{3n+8}\Big)
\end{aligned} \tag{130}
$$

$$
\begin{aligned}
&- v_7^2 \Big(\sum_{a+5=b} \phi_b^\dagger a \phi_a \, - \, \sum_{n=1}^{\infty} \phi_{3n+4}^\dagger (3n-1) \phi_{3n-1} \Big) \\
&- 2 v_7 v_1 \Big(\sum_{a=b+1} \phi_b^\dagger a \phi_a \, - \, \sum_{n=1}^{\infty} \phi_{3n-2}^\dagger (3n-1) \phi_{3n-1} \Big) \\
&- v_1^2 \Big(\sum_{a=b+7} \phi_b^\dagger a \phi_a \, - \, \sum_{n=1}^{\infty} \phi_{3n-2}^\dagger (3n+5) \phi_{3n+5} \Big) \\
&- G^3 \Big\{ \, 2(\phi_1 + \frac{v_1}{G})^3 \phi_6 \, + \, 12(\phi_1 + \frac{v_1}{G})^2 \phi_3 \phi_4 \, + \, 9(\phi_1 + \frac{v_1}{G}) \phi_2 \phi_3^2 \, + \, 8 \phi_2^3 \phi_3 \, \Big\} \\
&- \frac{G}{3} \sum_{n=1}^{\infty} \phi_{3n}^\dagger (3n+9) \phi_{3n+9} \, - \, 6 G^2 \phi_3 \phi_6 \, .
\end{aligned}
$$

Appendix 3

In this appendix we show that the number of fields which describe the Hamiltonian can be reduced by one, i.e. the Hamiltonian for (p,q) model is expressed by $p-1$ fields. We have explicitly checked this aspect for $p = 2, 3, 4$ cases. We here introduce the following new fields:

$$
\begin{aligned}
\tilde{\alpha}^{[i]}(z) \; &\stackrel{\text{def}}{\equiv} \; \alpha^{[i]}(z) - \alpha^{[0]}(z) \, , \\
\alpha^{[i]}(z) \; &= \; \sum_{n \in Z} \alpha_n^{[i]} z^{-n-1} \, .
\end{aligned} \tag{131}
$$

The current $\alpha^{[i]}(z)$ is written as

$$
\alpha^{[i]\star}(z) \; = \; \frac{1}{\sqrt{G}} \Big(v^{[p-i]}(z) + \phi^{[i]\dagger}(z) \Big) + \sqrt{G} \frac{\partial}{\partial z} \Big(z \phi^{[p-i]}(z) \Big) \, , \tag{132}
$$

where

$$
\begin{aligned}
\phi^{[i]\dagger}(z) \; &= \; \sum_{n=1}^{\infty} \phi_n^{[i]\dagger} z^{-n-1}, \qquad v^{[i]}(z) \; = \; \sum_{n=1}^{\infty} v_n^{[i]} z^{n-1}, \\
\phi^{[i]}(z) \; &= \; \sum_{n=1}^{\infty} \phi_n^{[i]} z^{n-1}.
\end{aligned} \tag{133}
$$

Here $\phi_n^{[i]\dagger}$, $\phi_n^{[i]}$ and so on are defined as the same as $\alpha_n^{[i]}$. The X operator is expressed by using this $\tilde{\alpha}^{[i]}$, for example,

$$
X \; = \; -\frac{1}{4} \oint \frac{dz}{2\pi i} : \frac{1}{3z^2} \tilde{\alpha}^{[1]3} \, + \, \frac{1}{4z^4} \tilde{\alpha}^{[1]} : \tag{134}
$$

for $p = 2$ case,

$$
X \; = \; -\frac{1}{108} \oint \frac{dz}{2\pi i} : \frac{1}{4z^6} \Big\{ 5(\tilde{\alpha}^{[1]4} + \tilde{\alpha}^{[2]4}) - 4 \tilde{\alpha}^{[1]} \tilde{\alpha}^{[2]} (\tilde{\alpha}^{[1]2} + \tilde{\alpha}^{[2]2}) \Big\} - \frac{1}{z^8} (\tilde{\alpha}^{[1]2} + \tilde{\alpha}^{[2]2}) : \tag{135}
$$

for $p=3$ case, and

$$\begin{aligned}
X &= -\frac{1}{4^4\cdot 6}\oint\frac{dz}{2\pi i}\frac{1}{z^{12}} : \frac{6}{5}(\tilde{\alpha}^{[1]5}+\tilde{\alpha}^{[3]5}-\tilde{\alpha}^{[2]5}) \\
&\qquad -2(\tilde{\alpha}^{[1]3}\tilde{\alpha}^{[3]2}+\tilde{\alpha}^{[1]2}\tilde{\alpha}^{[3]3})-5\tilde{\alpha}^{[2]2}(\tilde{\alpha}^{[1]3}+\tilde{\alpha}^{[3]3}) \\
&\qquad +3\tilde{\alpha}^{[2]2}(\tilde{\alpha}^{[1]2}\tilde{\alpha}^{[3]}+\tilde{\alpha}^{[1]}\tilde{\alpha}^{[3]2})+4\tilde{\alpha}^{[2]3}\tilde{\alpha}^{[1]}\tilde{\alpha}^{[3]} : \\
&\quad -\frac{1}{4^4\cdot 6}\oint\frac{dz}{2\pi i}\frac{1}{z^{14}} : 2(\tilde{\alpha}^{[1]3}+\tilde{\alpha}^{[3]3})-3\tilde{\alpha}^{[2]3}-6\tilde{\alpha}^{[1]}\tilde{\alpha}^{[3]}\tilde{\alpha}^{[2]} : \\
&\quad +\frac{9}{4^6}\oint\frac{dz}{2\pi i}\frac{1}{z^{16}}\tilde{\alpha}^{[2]} \qquad (136)
\end{aligned}$$

for $p=4$ case.

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